Random Variables

ECE 313

Probability with Engineering Applications

Lecture 8

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Today's Topics

- Review on Random Variables
- Cumulative Distribution Function (CDF)
- Probability Mass Function (PMF)
- Discrete Random Variables
  - Bernoulli
  - Binomial
  - Poisson
  - Geometric

Cumulative Distribution Function (CDF)

- Some properties of cdf F are:
  i. $F(b)$ is a non-decreasing function of b,
  ii. $\lim_{b \to +\infty} F(b) = F(\infty) = 1$,
  iii. $\lim_{b \to -\infty} F(b) = F(-\infty) = 0$.

- Property *(i)* follows since for $a < b$ the event $\{X \leq a\}$ is contained in the event $\{X \leq b\}$, and so it must have a smaller probability.
- Properties *(ii)* and *(iii)* follow since X must take on some finite value.

- All probability questions about X can be answered in terms of cdf $F(\cdot)$. For example:

$$P\{a \leq X \leq b\} = F(b) - F(a) \qquad \text{for all } a < b$$

i.e. calculate $P\{a \leq X \leq b\}$ by first computing the probability that $X \leq b$ $(F(b))$ and then subtract from this the probability that $X \leq a$ $(F(a))$.

Discrete Random Variables

- A random variable that can take on at most countable number of possible values is said to be *discrete.*
- For a discrete random variable X , we define the *probability mass function* $p(a)$ of X by:

$$p(a) = P\{X = a\}$$

- $p(a)$ is positive for at most a countable number of values of a .

 i.e., if X must assume one of the values *x1, x2, …,* then

$$p(x_i) > 0, \quad i = 1,2,...$$

$$p(x) = 0, \quad \text{for other values of x}$$

- Since take values *xi:*

$$\sum_{i=1}^{\infty} p(x_i) = 1$$

Cumulative Distribution Function

- The cumulative distribution function F can be expressed in terms of $p(a)$ by: $F(a) = \sum_{all\ x_i \leq a} p(x_i)$
- Suppose X has a probability mass function given by

$$p(1) = \frac{1}{2},\ \ p(2) = \frac{1}{3},\ \ p(3) = \frac{1}{6}$$

then the cumulative distribution function F of X is given by

$$F(a) = \begin{cases} 0, & a < 1 \\ \frac{1}{2}, & 1 \leq a < 2 \\ \frac{5}{6}, & 2 \leq a < 3 \\ 1 & 3 \leq a \end{cases}$$

Discrete Random Variables Examples

- Bernoulli
- Binomial
- Poisson
- Geometric

The Bernoulli Random Variable

$$p(0) = P\{X = 0\} = 1 - p,$$

$$p(1) = P\{X = 1\} = p$$

Where $p, 0 \leq p \leq 1$ is the probability that the trial is a success

X is said to be a *Bernoulli* random variable if its probability mass function is given by the above equation some for $p \in (0,1)$

The Binomial Random Variable

- *n* independent trials, each of which results in a "success" with *p* and in a "failure" with probability *1-p*
- If *X* represents the number of successes that occur in the *n* trials, *X* is said to be a binomial random variable with parameters (*n,p*)
- The probability mass function of a binomial random variable having parameters (*n,p*) is given by

$$p(i) = \binom{n}{i} p^i (1-p)^{n-i}, \qquad \text{Equation (2.3)}$$

where

$$\binom{n}{i} = \frac{n!}{(n-i)!\,i!}$$

- $\binom{n}{i}$ equals the number of different groups of *i* objects that can be chosen from a set of *n* objects

The Binomial Random Variable

- Equation (2.3) may be verified by first noting that the probability of any particular sequence of the *n* outcomes containing *i* successes and *n-i* failures is, by the assumed independence of trials, $p^i(1-p)^{n-i}$

- Equation (2.3) then follows since there are $\binom{n}{i}$ different sequences of the n outcomes leading to I successes and *n - i* failures. For instance if *n=3*, *i=2*, then there are $\binom{3}{2}=3$ ways in which the three trials can result in two successes.

- By the binomial theorem, the probabilities sum to one:

$$\sum_{i=0}^{\infty} p(i) = \sum_{i=0}^{n} \binom{n}{i} p^i (1-p)^{n-i} = (p + (1-p))^n = 1$$

Binomial Random Variable Example 1

- Four fair coins are flipped. Outcomes are assumed independent, what is the probability that two heads and two tails are obtained?

- Letting *X* equal the number of heads ("successes") that appear, then *X* is a binomial random variable with parameters *(n = 4, p = 12).* Hence by the binomial equation,

$$P\{X = 2\} = \binom{4}{2}(\frac{1}{2})^2(\frac{1}{2})^2 = \frac{3}{8}$$

Binomial Random Variable Example 2

- It is known that any item produced by a certain machine will be defective with probability 0.1, independently of any other item. What is the probability that in a sample of three items, at most one will be defective?

- If *X* is the number of defective items in the sample, then *X* is a binomial random variable with parameters (3, 0.1). Hence, the desired probability is given by:

$$P\{X=0\}+P\{X=1\}=\binom{3}{0}(0.1)^0(0.9)^3+\binom{3}{1}(0.1)^1(0.9)^2=0.972$$

Binomial RV Example 3

- Suppose that an airplane engine will fail, when in flight, with probability $1-p$ independently from engine to engine; suppose that the airplane will make a successful flight if at least 50 percent of its engines remain operative. For what values of p is a four-engine plane preferable to a two-engine plane?

- Because each engine is assumed to fail or function independently of what happens with the other engines, it follows that the number of engines remaining operative is a binomial random variable. Hence, the probability that a four-engine plane makes a successful flight is:

$$\binom{4}{2}p^2(1-p)^2+\binom{4}{3}p^3(1-p)+\binom{4}{4}p^4(1-p)^0$$

$$=6p^2(1-p)^2+4p^3(1-p)+p^4$$

Binomial RV Example 3 (Cont')

- The corresponding probability for a two-engine plane is:

$$\binom{2}{1} p(1-p) + \binom{2}{2} p^2 = 2p(1-p) + p^2$$

- The four-engine plane is safer if:

$$6p^2(1-p)^2 + 4p^3(1-p) + p^4 \geq 2p(1-p) + p^2$$

$$6p(1-p)^2 + 4p^2(1-p) + p^3 \geq 2 - p$$

$$3p^3 - 8p^2 + 7p - 2 \geq 0 \quad or \quad (p-1)^2(3p-2) \geq 0$$

- Or equivalently if: $3p - 2 \geq 0 \quad or \quad p \geq \dfrac{2}{3}$

- Hence, the four-engine plane is safer when the engine success probability is at least as large as 2/3, whereas the two-engine plane is safer when this probability falls below 2/3.

The Poisson Random Variable

- A random variable *X*, taking on one of the values 0,1,2,…, is said to be a *Poisson* random variable with parameter *λ*, if for some *λ*>0,

$$p(i) = P\{X = i\} = e^{-\lambda}\frac{\lambda^i}{i!}, \quad i = 0,1,\ldots$$

defines a probability mass function since

$$\sum_{i=0}^{\infty} p(i) = e^{-\lambda}\sum_{i=0}^{\infty}\frac{\lambda^i}{i!} = e^{-\lambda}e^{\lambda} = 1$$

The Poisson Random Variable Cont'd

- An important property of the Poisson random variable, it may be used to approximate a binomial random variable with the binomial parameter *n* is large and *p* is small

$$P\{X = i\} = \frac{n!}{(n-i)!\,i!} p^i (1-p)^{n-i}$$

$$= \frac{n!}{(n-i)!\,i!} \left(\frac{\lambda}{n}\right)^i \left(1 - \frac{\lambda}{n}\right)^{n-i}$$

$$= \frac{n(n-1)\cdots(n-i+1)}{n^i} \frac{\lambda^i}{i!} \frac{(1-\lambda/n)^n}{(1-\lambda/n)^i}$$

- Now, for *n* large and *p* small

$$\left(1 - \frac{\lambda}{n}\right)^n \approx e^{-\lambda}, \; \frac{n(n-1)\cdots(n-i+1)}{n^i} \approx 1, \left(1 - \frac{\lambda}{n}\right)^n \approx 1$$

- Hence for *n* large and *p* small

$$P\{X = i\} \approx e^{-\lambda} \frac{\lambda^i}{i!}$$

Continuous Random Variables

- Random variables whose possible variables whose set of possible values is uncountable

- *X* is a continuous random variable if there exists a nonnegative function *f(x)* defined for all real $x \in (-\infty, \infty)$, having the property that for any set of *B* real numbers

$$P\{X \in B\} = \int_B f(x)dx$$

- *f(x)* is called the *probability density function* of the random variable *X*

- The probability that *X* will be in *B* may be obtained by integrating the probability density function over the set *B*. Since *X* must assume some value, *f(x)* must satisfy

$$1 = P\{X \in (-\infty, \infty)\} = \int_{-\infty}^{\infty} f(x)dx$$

Continuous Random Variables Cont'd

- All probability statements about X can be answered in terms of $f(x)$
  e.g. letting $B=[a,b]$, we obtain $P\{a \le X \le b\} = \int_a^b f(x)dx$

- If we let $a=b$ in the preceding, then $P\{X = a\} = \int_a^a f(x)dx = 0$

- This equation states that the probability that a continuous random variable will assume any *particular* value is zero

- The relationship between the cumulative distribution $F(\cdot)$ and the probability density $f(\cdot)$

$$F(a) = P\{X \in (-\infty, a)\} = \int_{-\infty}^{a} f(x)dx$$

- Differentiating both sides of the preceding yields

$$\frac{d}{da}F(a) = f(a)$$

Continuous Random Variables Cont'd

- That is, the density of the derivative of the cumulative distribution function.

- A somewhat more intuitive interpretation of the density function

$$P\left\{a-\frac{\varepsilon}{2}\leq X\leq a+\frac{\varepsilon}{2}\right\}=\int_{a-\varepsilon/2}^{a+\varepsilon/2}f(x)dx\approx\varepsilon f(a)$$

when ε is small

- The probability that X will be contained in an interval of length ε around the point a is approximately $\varepsilon f(a)$