# Bernoulli Trials and TMR System

ECE 313

Probability with Engineering Applications

Lecture 7

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# Today's Topics

- Bernoulli Trials Examples
- TMR Systems
- Random Variables
- Cumulative Distribution Function (CDF)

# Bernoulli Trials Example 1

- Consider a binary communication channel transmitting coded words of n bits each. Assume that the probability of successful transmission of a single bit is p (and the probability of an error is q = 1-p), and the code is capable of correcting up to e (e >=0) errors.
- For example, if no coding or parity checking is used, then e = 0. If a single error correcting Hamming code is used then e =1.
- If we assume that the transmission of successive bits is independent, then the probability of successful word transmission is:

$$P_w = P(\text{"e or fewer errors in n trials"})$$

$$= \sum_{i=0}^{e} \binom{n}{i} (1-p)^i p^{n-i}$$

# Bernoulli Trials Example 2

- Consider a system with n components that requires m ($\leq$n) or more components to function for the correct operation of the system (called m-out-of-n system).

- If we let m=n, then we have a series system; if we let m = 1, then we have a system with parallel redundancy.
- Assume: n components are statistically identical and function independently of each other.

- Let R denote the reliability of a component ( and q = 1 - R gives its unreliability), then the experiment of observing the status of n components can be thought of as a sequence of n Bernoulli trials with the probability of success equal R.

# Bernoulli Trials Example 2 (cont.)

- Now the reliability of the system is:

$$R_{m|n} = P(\text{"}m \text{ or more components functioning properly"})$$

$$= P\left(\bigcup_{i=m}^{n} \{\text{"exactly } i \text{ components functioning properly"}\}\right)$$

$$= \sum_{i=m}^{n} P(\text{"exactly } i \text{ components functioning properly"})$$

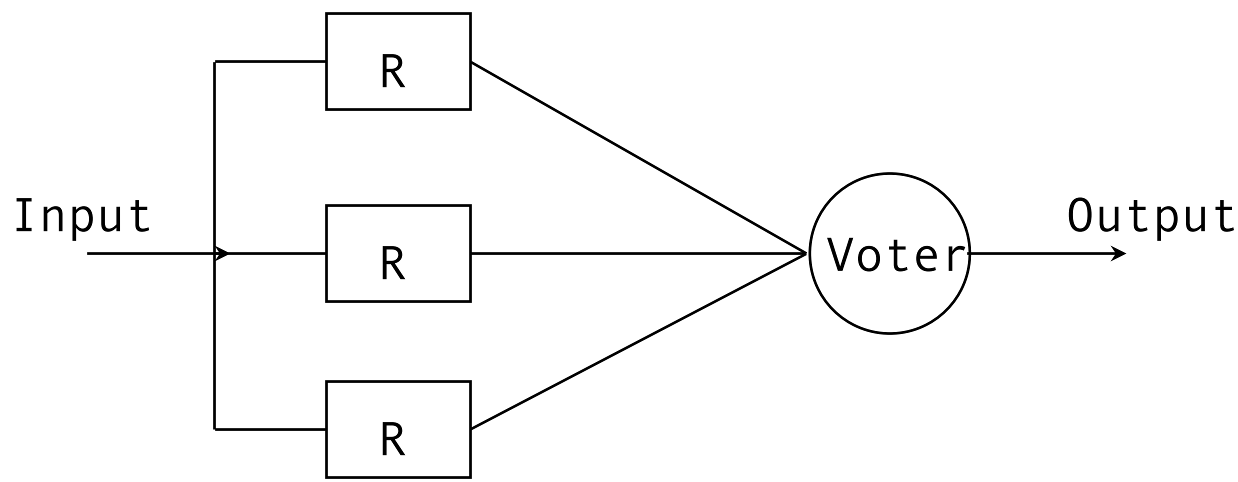
$$= \sum_{i=m}^{n} p(i) = \sum_{i=m}^{n} \binom{n}{i} R^{i} (1-R)^{n-i}$$

- It is easy to verify that: $R_{1|n} = R(parallel) = 1-(1-R)^n \quad and$

$$R_{n|n} = R(series) = R^n$$

# Bernoulli Trials
## TMR System Example

- As special case of m-out-of-n system, consider a system with triple modular redundancy (TMR). In such a system there are three components, two of which are required to be in working order for the system  to function properly (i.e., n = 3 and m = 2). This is achieved by feeding the outputs of the three components into a majority voter.



# Bernoulli Trials
## TMR System Example (cont.)

- The reliability of TMR system is given by the expression:

$$R_{TMR} = \sum_{i=2}^{3} \binom{3}{i} R^i (1-R)^{3-i} = \binom{3}{2} R^2 (1-R) + \binom{3}{3} R^3 (1-R)^0 = 3R^2(1-R) + R^3$$

- and thus

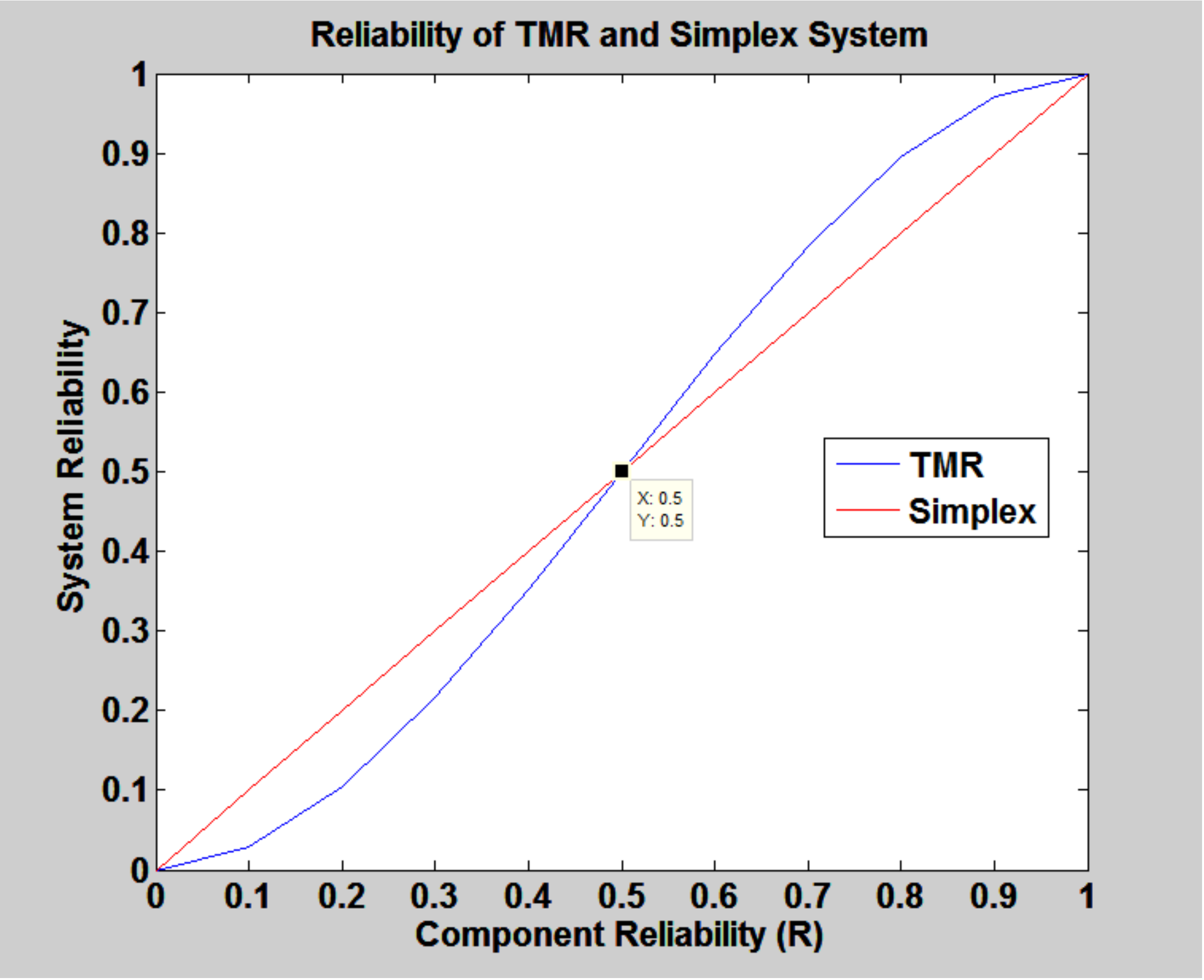
$$R_{TMR} = 3R^2 - 2R^3$$

<u>Note that</u>:

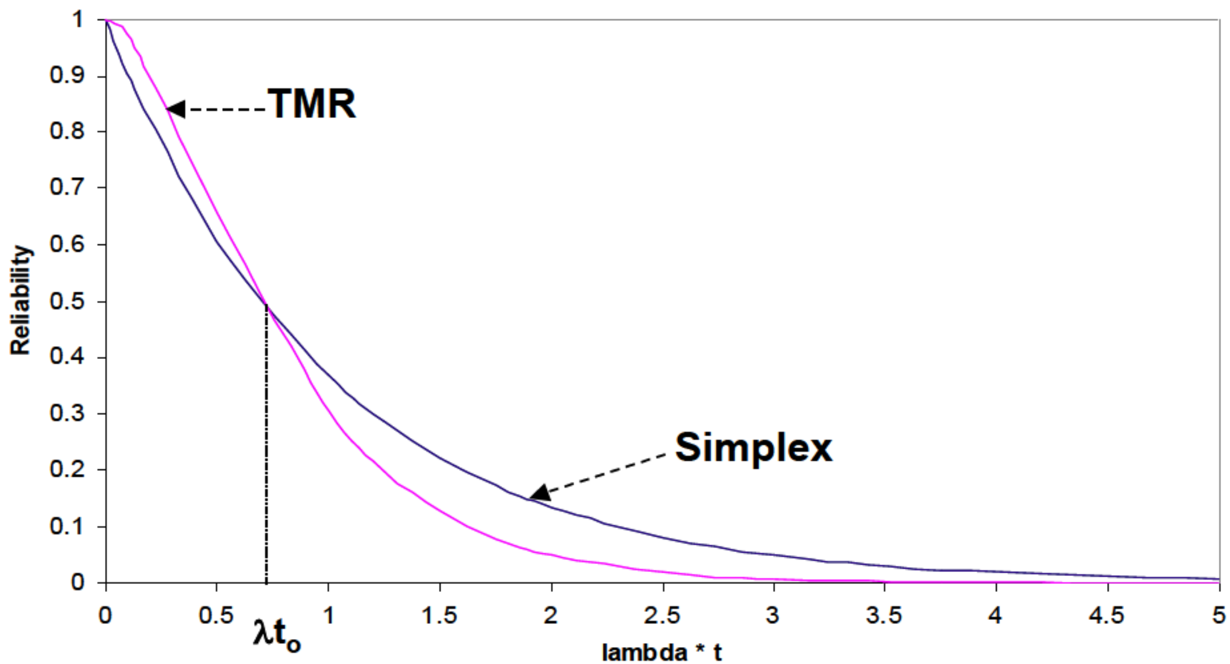
$$R_{TMR} \begin{cases} > R, & if\, R > \tfrac{1}{2} \\ = R, & if\, R = \tfrac{1}{2} \\ < R, & if\, R < \tfrac{1}{2} \end{cases}$$

- Thus TMR increases reliability over the simplex system only if the simplex reliability is greater than 0.5; otherwise decreases reliability
- Note: the voter output corresponds to a majority; it is possible for two or more malfunctioning units to agree on an erroneous vote.

# Reliability of TMR vs. Simplex



# Reliability of TMR vs. Simplex




$$R_{TMR}(t) \geq R(t) \qquad 0 \leq t \leq t_0$$

$$R_{TMR}(t) \leq R(t) \qquad t_0 \leq t < \infty$$

$$\text{where } t_0 = \frac{\ln 2}{\lambda} \approx \frac{0.7}{\lambda}$$

# Random Variable

- Definition: Random Variable

A random variable X on a sample space S is a function $X: S \rightarrow \Re$ that assigns a real number X(s) to each sample point $s \in S$.

Example: Consider a random experiment defined by a sequence of three Bernoulli trials. The sample space S consists of eight triples (where 1and 0 respectively denote success and a failure on the nth trail). The probability of successes , p, is equal 0.5.

| Sample points | P(s) | X(s) |
|---|---|---|
| 111 | 0.125 | 3 |
| 110 | 0.125 | 2 |
| 101 | 0.125 | 2 |
| 100 | 0.125 | 1 |
| 011 | 0.125 | 2 |
| 010 | 0.125 | 1 |
| 001 | 0.125 | 1 |
| 000 | 0.125 | 0 |

Note that two or more sample points might give the same value for X (i.e., X may not be a one-to-one function.), but that two different numbers in the range cannot be assigned to the same sample point (i.e., X is well defined function).

# Random Variable (cont.)

- *Event space*

  For a random variable X and a real number x, we define the event $A_x$ to be the subset of S consisting of all sample points s to which the random variable X assigns the value x.

  $A_x = \{s \in S \mid X(s) = x\}$; Note that: $\bigcup_{x \in \Re} A_x = S$

  The collection of events $A_x$ for all x defines an *event space*

- In the previous example the random variable defines four events:

  $A_0 = \{s \in S \mid X(s) = 0\} = \{(0, 0, 0)\}$

  $A_1 = \{(0, 0, 1), (0, 1, 0), (1, 0, 0)\}$

  $A_2 = \{(0, 1, 1), (1, 0, 1), (1, 1, 0)\}$

  $A_3 = \{(1, 1, 1)\}$

**Discrete random variable**

The random variable which is either finite or countable.

# Random Variables Example 1

- Let $X$ denote the random variable that is defined as the sum of two fair dice; then

$$P\{X=2\} = P\{(1,1)\} = \frac{1}{36},$$

$$P\{X=3\} = P\{(1,2),(2,1)\} = \frac{2}{36},$$

$$P\{X=4\} = P\{(1,3),(2,2),(3,1)\} = \frac{3}{36},$$

$$\vdots$$

$$P\{X=9\} = P\{(3,6),(4,5),(5,4),(6,3)\} = \frac{4}{36},$$

$$P\{X=10\} = P\{(4,6),(5,5),(6,4)\} = \frac{3}{36},$$

$$P\{X=11\} = P\{(5,6),(6,5)\} = \frac{2}{36},$$

$$P\{X=12\} = P\{(6,6)\} = \frac{1}{36}$$

# Random Variables Example 1 (Cont'd)

- In other words, the random variable *X* can take on any integral value between two and twelve, and the probability that it takes on each value is given.

- Since *X* must take on one of the values two through twelve, we must have:

$$1 = P\left\{ \bigcup_{i=2}^{12} \{X = n\} \right\} = \sum_{n=2}^{12} P\{X = n\}$$

which may be checked from the previous equations.

# Random Variables Example 2

- Suppose that our experiment consists of tossing two fair coins. Letting *Y* denote the number of heads appearing, then *Y* is a random variable taking on one of the values 0, 1, 2 with respective probabilities:

$$P\{Y=0\}=P\{(T,T)\}=\frac{1}{4},$$

$$P\{Y=1\}=P\{(T,H),(H,T)\}=\frac{2}{4},$$

$$P\{Y=2\}=P\{(H,H)\}=\frac{1}{4}$$

$$P\{Y=0\}+P\{Y=1\}+P\{Y=2\}=1.$$

# Random Variables Example 3

- Suppose that we toss a coin having a probability *p* of coming up heads, until the first head appears. Letting *N* denote the number of flips required, then assuming that the outcome of successive flips are independent, *N* is a random variable taking on one of the values 1, 2, 3, . . . , with respective probabilities

$$P\{N=1\} = P\{H\} = p,$$
$$P\{N=2\} = P\{(T,H)\} = (1-p)p,$$
$$P\{N=3\} = P\{(T,T,H)\} = (1-p)^2 p,$$
$$\vdots$$
$$P\{N=n\} = P\{(\underbrace{T,T,....,T}_{n-1},H)\} = (1-p)^{n-1} p, \qquad n \geq 1$$

# Random Variables Example 3 (Cont'd)

- As a check, note that

$$P\left(\bigcup_{n=1}^{\infty}\{N=n\}\right)=\sum_{n=1}^{\infty}P\{N=n\}$$

$$=p\sum_{n=1}^{\infty}(1-p)^{n-1}$$

$$=\frac{p}{1-(1-p)}$$

$$=1$$

# Random Variables Example 4

- Suppose that our experiment consists of seeing how long a battery can operate before wearing down. Suppose also that we are not primarily interested in the actual lifetime of the battery but are concerned only about whether or not the battery lasts at least two years. In this case, we may define the random variable *I* by

$$I = \begin{cases} 1, & \text{if the lifetime of battery is two or more years} \\ 0, & \text{otherwise} \end{cases}$$

- If *E* denotes the event that the battery lasts two or more years, then the random variable *I* is known as the *indicator* random variable for event *E*. (Note that *I* equals 1 or 0 depending on whether or not *E* occurs.)

# Random Variables Example 5

- Suppose that independent trials, each of which results in any of *m* possible outcomes with respective probabilities *p*1, . . . , *pm*, $\sum_{i=1}^{m} p_i = 1$ are continually performed. Let *X* denote the number of trials needed until each outcome has occurred at least once.
- Rather than directly considering *P*{*X* = *n*} we will first determine *P*{*X* > *n*}, the probability that at least one of the outcomes has not yet occurred after *n* trials. Letting *Ai* denote the event that outcome *i* has not yet occurred after the first *n* trials, *i* = 1,...,*m*, then:

$$P\{X > n\} = P\left(\bigcup_{i=1}^{m} A_i\right)$$

$$= \sum_{i=1}^{m} P(A_i) - \sum\sum_{i<j} P(A_i A_j)$$

$$+ \sum\sum_{i<j<k}\sum P(A_i A_j A_k) - .... + (-1)^{m+1} P(A_1 ... A_m)$$

# Random Variables Example 5 (Cont'd)

- Now, $P(A_i)$ is the probability that each of the first *n* trials results in a non-*i* outcome, and so by independence

$$P(A_i) = (1 - p_i)^n$$

- Similarly, $P(A_i A_j)$ is the probability that the first *n* trials all result in a non-*i* and non-*j* outcome, and so

$$P(A_i A_j) = (1 - p_i - p_j)^n$$

- As all of the other probabilities are similar, we see that

$$P\{X > n\} = \sum_{i=1}^{m} (1 - p_i)^n - \sum\sum_{i<j} (1 - p_i - p_j)^n$$

$$+ \sum\sum_{i<j<k}\sum (1 - p_i - p_j - p_k)^n - \ldots$$

# Random Variables Example 5 (Cont'd)

- Since $P\{X = n\} = P\{X > n-1\} - P\{X > n\}$
- By using the algebraic identity: $(1-a)^{n-1} - (1-a)^n = a(1-a)^{n-1}$
- We see that:

$$P\{X = n\} = \sum_{i=1}^{m} p_i (1-p_i)^{n-1} - \sum_{i<j}\sum (p_i + p_j)(1 - p_i - p_j)^{n-1}$$

$$+ \sum\sum_{i<j<k}\sum (p_i + p_j + p_k)(1 - p_i - p_j - p_k)^{n-1} - \ldots$$

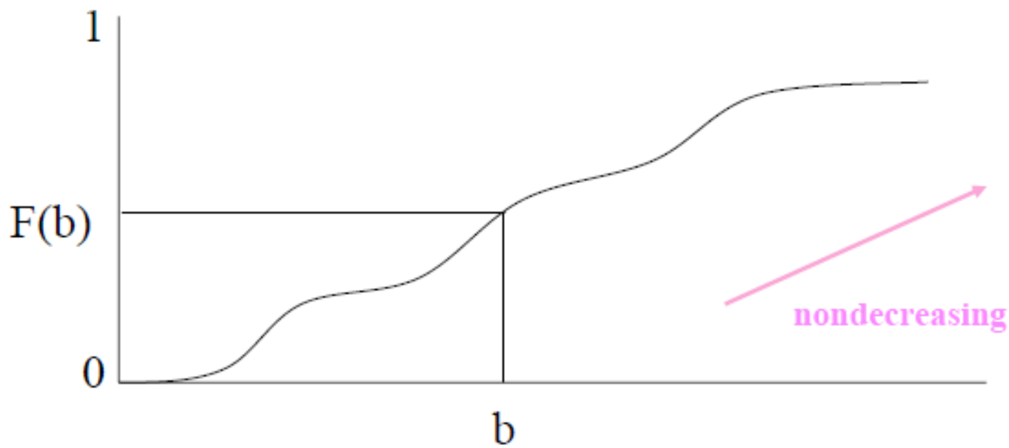
# Discrete/Continuous Random Variables

- So far the random variables of interest were either a finite or a countable number of possible values (*discrete* random variables). Random variables can also take on a continuum of possible values (known as *continuous* random variables).
- Example: A random variable denoting the lifetime of a car, when the car's lifetime is assumed to take on any value in some interval *(a,b).*

# Cumulative Distribution Function (CDF)

- The *cumulative distribution function* (cdf) (or *distribution function*) $F(\cdot)$ of a random variable $X$ is defined for any real number $b, -\infty < b < \infty$, by $F(b) = P\{X \leq b\}$

- $F(b)$ denotes the probability that the random variable $X$ takes on a value that is less than or equal to $b$.



# Cumulative Distribution Function (CDF)

- Some properties of cdf $F$ are:
  **i.** $F(b)$ is a non-decreasing function of b,
  **ii.** $\lim_{b\to+\infty} F(b) = F(\infty) = 1,$
  **iii.** $\lim_{b\to-\infty} F(b) = F(-\infty) = 0.$

- Property *(i)* follows since for $a < b$ the event $\{X \le a\}$ is contained in the event $\{X \le b\}$, and so it must have a smaller probability.
- Properties *(ii)* and *(iii)* follow since $X$ must take on some finite value.

- All probability questions about $X$ can be answered in terms of cdf $F(\cdot)$. For example:

$$P\{a \le X \le b\} = F(b) - F(a) \qquad \text{for all } a < b$$

i.e. calculate $P\{a \le X \le b\}$ by first computing the probability that $X \le b$ $(F(b))$ and then subtract from this the probability that $X \le a$ $(F(a))$.