Reliability Applications (Independence and Bayes Rule)

ECE 313

Probability with Engineering Applications

Lecture 5

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Today's Topics

- Review of Physical vs. Stochastic Independence
  - Example
- Reliability Applications
  - Multiversion Programming
    - N-version software
    - Recovery blocks
  - Series and Parallel Systems

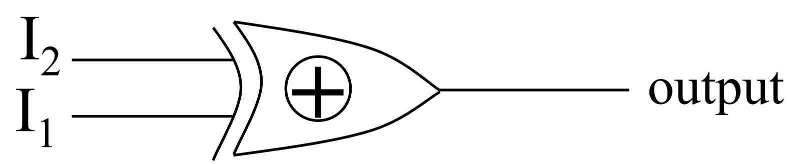
Physical vs. Stochastic Independence Example

- Stochastic independence does not imply physical independence.
- Example:

$A(I_1=1)$

$B(I_2=1)$

C (output=1)



- Assume that A,B are independent and that P(A)=P(B)=1/2
- C => event, output = 1
- Questions: Are A and C independent? Explain

Physical vs. Stochastic Independence Example

- $P(A) = 1/2$
- $P(C) = 1/2$
- $P(A \cap C) = 1/4 = P(A).P(C)$

⇒ A and C are stochastically independent.

⇒ But physically they are not independent, because there is a logical relation between the inputs and the output of a XOR gate and Event C (output=1) happens when events A (I_1=1) and $\overline{B}$ (I_2=0) happen.

Multiversion Programming

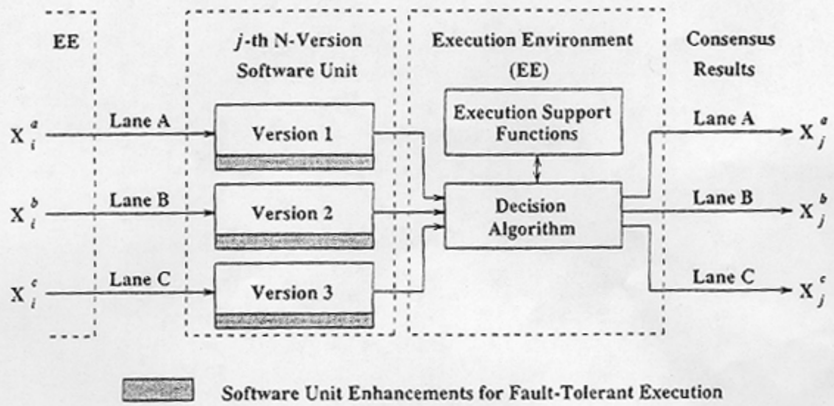


Figure 2.1 The N-version software (NVS) model with n = 3

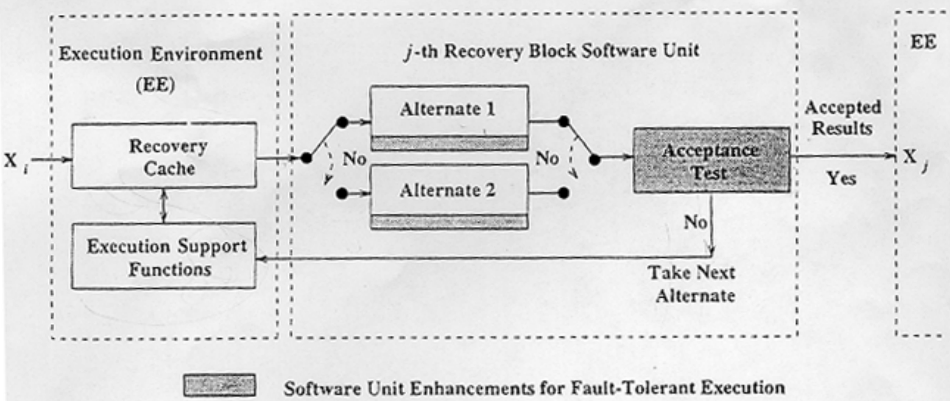
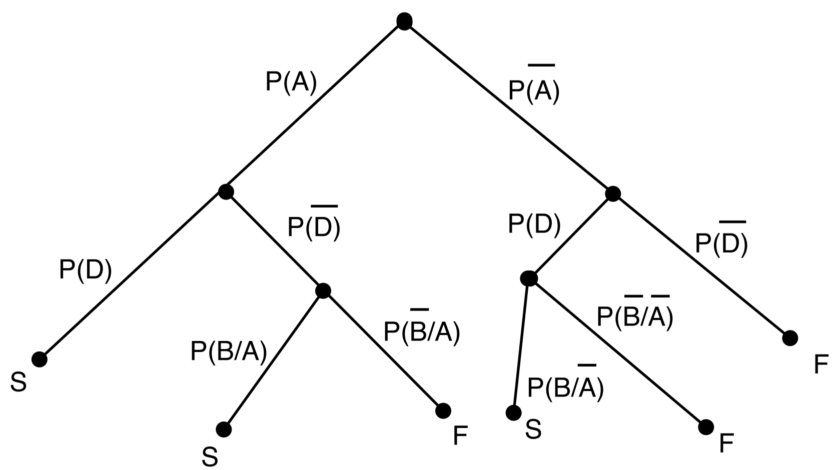


Figure 2.2 The recovery block (RB) model

Multiversion Programming

- A technique for fault-tolerant software [Randell 1978], consists of a primary and an alternate module for each critical task and a test for determining whether a module performed its function correctly. This is called a **recovery block**.
- If the primary module fails, the detection test indicates its failure and the alternate module is activated.
- Define the events:
  - A = "Primary module functions correctly."
  - B = "Alternate module functions correctly."
  - D = " Detection test following the execution of the primary performs its task correctly."
- Assume that event pairs *A* and *D* as well as *B* and *D* are independent but events *A* and *B* are dependent. Derive an expression for the failure probability of a recovery block.

Multiversion Programming (cont.)



$$P(\text{failure}) = P(\overline{B} \mid A)\, P(\overline{D})\, P(A) + P(\overline{B} \mid \overline{A})\, P(D) P(\overline{A}) + P(\overline{A})\, P(\overline{D})$$

$$= P(\overline{B} \cap A)\, P(\overline{D}) + P(\overline{B} \cap \overline{A})\, P(D) + P(\overline{A})\, P(\overline{D})$$

Distributive Law (E3)

$$= P((\overline{B} \cap A) \cup \overline{A})\, P(\overline{D}) + P(\overline{B} \cap \overline{A})\, P(D)$$

$$= P((\overline{B} \cup \overline{A}) \cap (A \cup \overline{A})) P(\overline{D}) + P(\overline{B} \cap \overline{A})\, P(D)$$

$$= P(\overline{B} \cup \overline{A})\, P(\overline{D}) + P(\overline{B} \cap \overline{A})\, P(D)$$

Application to Reliability Evaluation

- Consider the problem of computing reliability of so-called series-parallel systems.
- A **series system** is one in which all components are so interrelated that the entire system will fail if any one of its components fails.
- A **parallel system** is one that will fail only if all its components fail.
- We will assume that failure events of components in a system are mutually independent. Consider a series system of n components.

Application to Reliability Evaluation (cont.)

- For I=1,2,…,n, define events A_i="Component i is functioning properly." The **reliability**, R_i, of component i is defined as the probability that the component is functioning properly. Then: $R_i=P(A_i)$.
- By the assumption of series connections, the system reliability:

$$
\begin{aligned}
R_s &= P\,(\text{"The system is functioning properly."}) \\
&= P(A_1 \cap A_2 \cdots \cap A_n) \\
&= P(A_1)P(A_2)\cdots P(A_n) \\
&= \prod_{i=1}^{n} R_i \qquad (2.1)
\end{aligned}
$$

Example of Effect of Complexity on Reliability

- This example demonstrates how quickly system reliability degrades with an increase in complexity.
- For example, if a system consists of five components in a series, each having a reliability of 0.970, then the system reliability is $0.970^5=0.859$.
- If the system complexity is increased so that it contains ten similar components, its reliability is reduced $0.970^{10}=0.738$.
- Consider what happens to system reliability when a large system such as a computer system consists of tens to hundreds of thousands of components.

Increasing Reliability Using Redundancy

- One way to increase the reliability of a system is to use redundancy, i.e., to replicate components with small reliabilities. This is called **parallel redundancy**.
- Consider a system consisting of *n* independent components in parallel; the system fails to function only if *n* components have failed.
- Define event A_i="The component *i* is functioning properly" and

  A_p="The parallel system of *n* components is functioning properly."

  Also let R_i=P(A_i) and R_p=P(A_p).

  To establish a relation between A_p and the A_i's, it is easier to consider the complementary events.

  Thus: $\overline{A}_p$ = "The parallel system has failed."

  = "All *n* components have failed."

  $= \overline{A}_1 \cap \overline{A}_2 \cap \cdots \cap \overline{A}_n$

- Therefore, by independence:

$$P(\overline{A}_p) = P(\overline{A}_1 \cap \overline{A}_2 \cap \cdots \cap \overline{A}_n) = P(\overline{A}_1)P(\overline{A}_2)\cdots P(\overline{A}_n)$$

Increasing Reliability Using Redundancy (cont.)

- Now let F_p=1-R_p be the unreliability of the parallel system, and similarly let F_i=1-R_i be the unreliability of component *i*.
- Then, since A_i and $\overline{A}_i$ are mutually exclusive and collectively exhaustive events, we have: $1 = P(S) = P(A_i) + P(\overline{A}_i)$
- And: $F_i = P(\overline{A}_i) = 1 - P(A_i)$
- Then: $F_p = P(\overline{A}_i) = \prod_{i-1}^{n} F_i$
- And: $R_p = 1 - F_p = 1 - \prod_{i=1}^{n}(1 - R_i)$

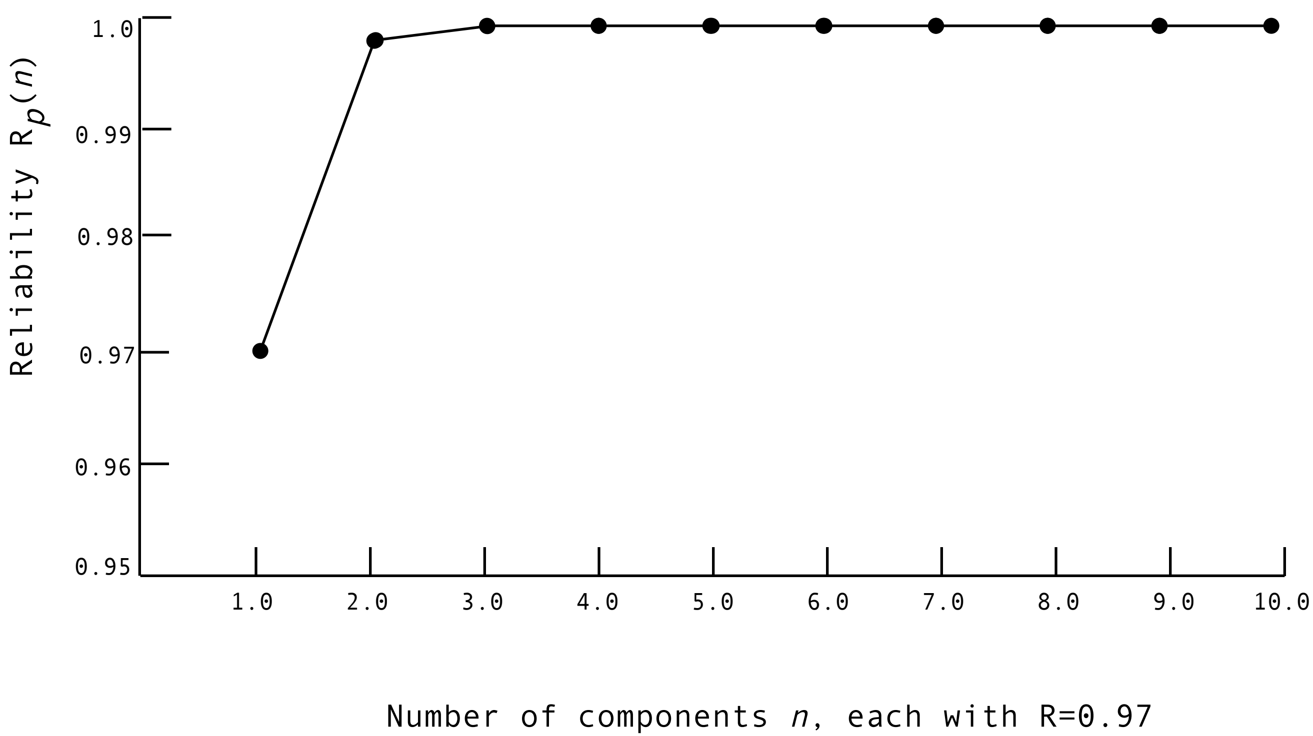
Product Law of Unreliabilities

- Thus, for parallel systems of *n* independent components, we have a **product law of unreliabilities** analogous to the **product law of reliabilities** of series systems.
- If we have a parallel system of five components, each with a reliability of 0.970, then the system reliability is increased to:

$$1-(1-0.970)^5 = 1-(0.03)^5 = 1-0.0000000243 = 0.9999999757$$

- However, we should be aware of the **law of diminishing returns**.

Reliability Curve of a Parallel Redundant System



Reliability of Series-Parallel Systems

- We can use formulas parallel and series systems in combination to compute the reliability of a system having both series and parallel parts (**series-parallel systems**).
- Consider a series-parallel system of n serial stages, where stage *i* consists of n_i identical components in parallel.

Reliability of Series-Parallel Systems (cont.)

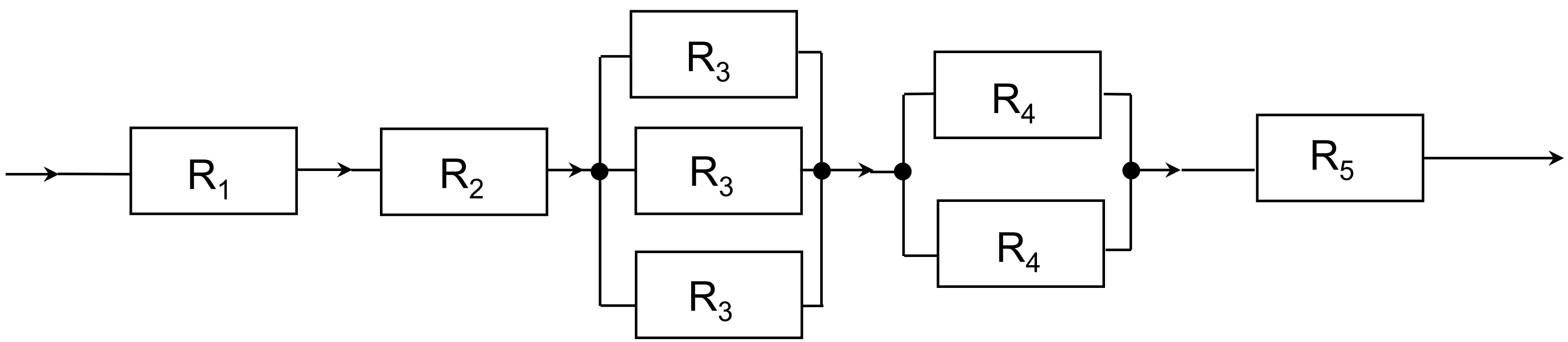
- The reliability of each component of stage *i* is R_i.
- Assuming that all components are independent, R_{sp} :

$$R_{sp} = \prod_{i=1}^{n} [1 - (1 - R_i)^{n_i}]$$

Series-Parallel System Example

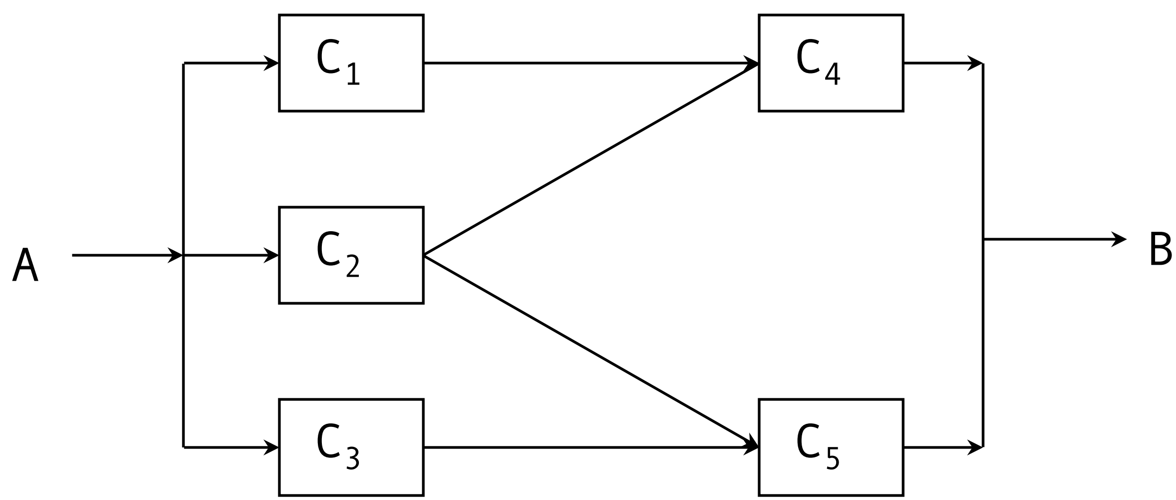
- Consider the series-parallel system shown in the diagram below, consisting of five stages, with $n_1=n_2=n_5=1$, $n_3=3$, $n_4=2$, and $R_1=0.95$, $R_2=0.99$, $R_3=0.70$, $R_4=0.75$, and $R_5=0.9$.

- Then: $R_{sp} = 0.95 \cdot 0.99 \cdot (1-0.3^3) \cdot (1-0.25^2) \cdot 0.9 = 0.772$



Application of Bayes
Complex system Example

- Consider evaluating the reliability R of the five-component system. The system is said to be functioning properly only if all the components on at least one path from point A to point B are functioning properly.



Bayes' Formula Example 3 (cont.)

- Define for i = 1, 2, …, 5 event X_i = "Component i is functioning properly"
  - let R_i = reliability of component i = $P(X_i)$
  - let X = "System is functioning properly
  - let R = system reliability = P(X)
- Thus X is a union of four events

$$X = (X_1 \cap X_4) \cup (X_2 \cap X_4) \cup (X_2 \cap X_5) \cup (X_3 \cap X_5)$$

- These four events are not mutually exclusive. Therefore, we cannot directly use axiom (A3). Note, however, that we could use relation (Rd), which does apply to union of interesting events. But this method is computationally tedious for a relatively long list of events. Instead, using the theorem of total probability, we have:

$$P(X) = P(X \cap X_2) + P(X \cap \overline{X_2})$$

$$P(X) = P(X \mid X_2)P(X_2) + P(X \mid \overline{X}_2)P(\overline{X}_2) = P(X \mid X_2)R_2 + P(X \mid \overline{X}_2)(1 - R_2)$$

Bayes' Formula Example 3 (cont.)

- Now to compute $P(X|X_2)$ observe that since component C_2 is functioning, the status of components C_1 and C_3 is irrelevant. The system is equivalent to two components C_4 and C_5 in parallel. Therefore we get: $P(X \mid X_2) = 1-(1-R_4)(1-R_5)$

- To compute $P(X|\overline{X}_2)$, since C_2 is known to have failed, the resulting system is a series-parallel one whose reliability is:

$$P(X \mid \overline{X}_2) = 1-(1-R_1R_4)(1-R_3R_5)$$

- Combining previous equations and substituting, we have:

$$P(x) = R = [1-(1-R_4)(1-R_5)]R_2 + [1-(1-R_1R_4)(1-R_3R_5)](1-R_2)$$
$$= 1 - R_2(1-R_4)(1-R_5) - (1-R_2)(1-R_1R_4)(1-R_3R_5)$$