Bayes' Formula
Independence of Events

ECE 313

Probability with Engineering Applications

Lecture 4

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Today's topics

- Review conditional probabilities
  => Theorem of total probabilties
- Bayes formula (Rule)
- Independence of Events
- Examples

Conditional Probability

- The *probability of A given B (P(A|B))* defines the conditional probability of the event A given that the event B has occured and is given by:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

  if $P(B) \neq 0$ and is undefined otherwise.

- A rearrangement of the above definition gives the following *multiplication rule (MR)*

$$P(A \cap B) = \begin{cases} P(B)P(A \mid B) & \text{if } P(B) \neq 0 \\ P(A)P(B \mid A) & \text{if } P(A) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

Theorem of Total Probability

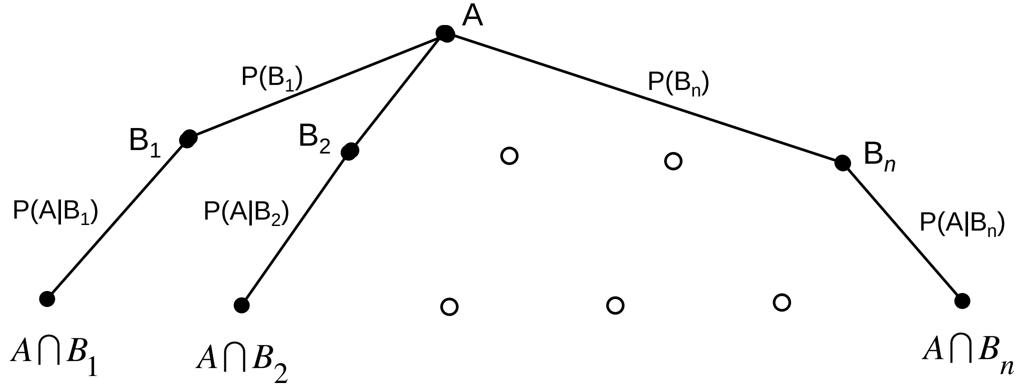
- An event B (probability P(B)) partitions a sample space S into two disjoint subsets B and $\overline{B}$ (exhaustive and exclusive)
- Consider S' ={B,$\overline{B}$} with associated probabilities P(B) and P($\overline{B}$), S' (**event space**).
- If A is another event in S', then: $A = (A \cap B) \cup (A \cap \overline{B})$.
- Then: $P(A) = P(A \cap B) + P(A \cap \overline{B})$
- And, using the definition of **conditional probability**, this equals:

$$P(A \mid B)P(B) + P(A \mid \overline{B})P(\overline{B})$$

Theorem of Total Probability

- This relation can be generalized with respect to the event space S' ={B_1,B_2,…,B_n) where B_1,B_2,…B_n are collectively exhaustive and mutually exclusive:

$$P(A) = \sum_{i=1}^{n} P(A \mid B_i) P(B_i)$$



The Theorem of Total Probability

- The product of all probabilities from the root of the tree to any node equals the probability of the event represented by that node. P(A) can be computed by summing probabilities associated with all the leaf nodes of the tree.

Bayes' Formula

- The situation often arises in which event A has occurred, but it is not known which of the events $B_1, B_2, \ldots B_n$ has occurred.
- To evaluate $P(B_j|A)$ (the conditional probability that one of the events B_j occurs given that A occurs), by the definition of conditional probability and the theorem of total probability:

$$P(B_j \mid A) = \frac{P(B_j \cap A)}{P(A)} = \frac{P(A \mid B_j)P(B_j)}{\sum_i P(A \mid B_i)P(B_i)}$$

- Bayes' Formula is useful in many applications and forms the basis of a statistical method called **bayesian procedure**.

Bayes' Rule Example 1

- Measurements at NCSA's Blue Waters Supercomputer at the University of Illinois indicated that the source of incoming jobs is
  - 15% from Industry
  - 35% from UIUC, and
  - 50% from the Great Lakes Consortium.

- Suppose that some jobs initiated from each of these sites requires a system configuration change (a set-up time). The set-up probabilities are 0.01, 0.05, and 0.02 respectively.

- Find the probability that a job chosen at random at NCSA's Blue Waters system is a ***set-up job***. Also find the probability that a randomly chosen job comes from UIUC, given that it is a *set-up job*.

Bayes' Rule Example 1 (cont.)

- Define events B_i="Job is from site i" (i=1, 2, 3 for Industry, UIUC, Great Lakes Consortium, respectively) and A="Job requires set-up." Then by the theorem of total probability:

$$P(A) = P(A \mid B_1)P(B_1) + P(A \mid B_2)P(B_2) + P(A \mid B_3)P(B_3)$$
$$= (0.01)\cdot(0.15) + (0.05)\cdot(0.35) + (0.02)\cdot(0.5) = 0.029$$

- Now the second event of interest is $[B_2 \mid A]$, and from Bayes' rule:

$$P(B_2 \mid A) = \frac{P(A \mid B_2)P(B_2)}{P(A)} = \frac{0.05\cdot 0.35}{0.029} = 0.603$$

- The knowledge that the job is multi-tasking increases the probability that it came from UIUC from 35 percent to 60 percent

Example 2

- We are given a box containing 5,000 IC chips, of which 1,000 are manufactured by company X and the rest by company Y. Ten percent of the chips made by company X and 5 percent of the chips made by company Y are defective. If a randomly chosen chip is found to be defective, find the probability that it came from company X.

Given that the chip is defective, it came from company X.

P(A) = chip came from X.

P(B) = chip is defective.

Example 3 (cont.)

- Define events A = "Chip is made by company X" and

  B = "Chip is defective."

  P(A) = 1,000/5,000 = 0.2

  (out of a total of 5,000 chips, 300 are defective)

  P(B) = 300/5,000 = 0.06

  Event $A \cap B$ = "Chip is made by company X and is defective"
- Out of 5,000 chips, 100 chips qualify for this statement
- Thus $P(A \cap B)$ = 100/5,000 = 0.02
- Now: $$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{0.02}{0.06} = \frac{1}{3}$$

Example 3 (cont.)

- Note: Knowledge of occurrence of Event B has increased the probability of occurrence of Event A. Similarly show that knowledge of occurrence of A has increased the chances of occurrence of B, (P(B/A) = 0.1)

$$\frac{P(A/B)}{P(B/A)} = \frac{1/3}{0.1} = \frac{0.2}{0.06} = \frac{P(A)}{P(B)}$$

- This property of conditional probabilities holds in general:

$$\frac{P(A/B)}{P(B/A)} = \frac{P(A \cap B)/P(B)}{P(A \cap B)/P(A)} = \frac{P(A)}{P(B)}$$

- *Repeat this example with A = "Chip is made by company Y"*

Independence of Events

- We define two events A and B to be independent if and only if:

  P(A|B)=P(A)

- From the definition of conditional probability [provided P(A) $\neq$ 0 and P(B)) $\neq$ 0]:

  $$P(A \bigcap B) = P(A)P(A \mid B) = P(B)P(A \mid B)$$

- This leads to the following usual definition of independence:

  Events A and be are said to be independent if:

  $$P(A \bigcap B) = P(A)P(B)$$

  Such events are also referred to as "stochastically independent events" or "statistically independent events."

- Note: If A and B are not independent, then $P(A \bigcap B)$ is computed using the multiplication rule.

Example

- Consider the experiment of tossing two dice. The sample space is S = {(i,j)|1 ≤ i,j ≤ 6}. Assume all the sample points have the equal probability of 1/36. Let:

  A = "The first die results in a 1, 2, or 3."

  B = "The second die results in a 4, 5, or 6."

  C = "The sum of the two faces is 7."

- Then : $A \cap B = \{(1,4),(1,5),(1,6),(2,4),(2,5),(2,6),(3,4),(3,5),(3,6)$
- And: $A \cap C = B \cap C = A \cap B \cap C = \{(1,6),(2,5)(3,4)\}$
- Therefore: $P(A \cap B) = 1/4 = P)A)P(B)$

  $$P(A \cap C) = 1/12 = P(A)P(C)$$
  $$P(B \cap C) = 1/12 = P(B)P(C)$$

- But: $P(A \cap B \cap C) = 1/12 \neq P(A)P(B)P(C) = 1/24$
- In this example, events A, B, and C are pairwise independent but not mutually independent.

Example (cont.)

- If the events A1, A2, …, An are such that every pair is independent, then they are called **pairwise independent**. It does not follow that the list of events is **mutually independent**.

- *Repeat this example with C = "The sum of the two faces is 9."*

Some Important Points about the Concept of Independence

- If A and B are two mutually exclusive events, then $A \cap B = \varnothing$, which implies $P(A \cap B) = 0$. Now, if they are independent as well, then either P(A)=0 or P(B)=0.

- If the events A and B are independent, and the events B and C are independent, then events A and C need not be independent (i.e., independence is not a transitive relation).

Some Important Points about the Concept of Independence (cont.)

- If the events A and B are independent, then so are events $\overline{A}$ and B, events A and $\overline{B}$, and events $\overline{A}$ and $\overline{B}$. Note that $A \cap B$ and $\overline{A} \cap B$ are mutually exclusive events whose union is B, i.e.,

$$P(B) = P(A \cap B) + P(\overline{A} \cap B) = P(A)P(B) + P(\overline{A} \cap B),$$

since A and B are independent.

This implies

$$P(\overline{A} \cap B) = P(B) - P(A)P(B) = P(B)[1 - P(A)] = P(B)(P(\overline{A}).$$

- The independence of A and $\overline{B}$ and $\overline{A}$ and $\overline{B}$ can be shown similarly.

- The concept of independence of two events can be extended to a list of *n* events.

Physical vs. Stochastic Independence

- It may be reasonable to assume that two events are physically independent.
- Example: Coin tossing (One toss does not influence another.)
- Other examples:
  - Arrivals of jobs to a computer system
  - Disk access
  - Phone calls arriving at an exchange
- Physical independence is usually used to assert stochastic independence.
- This assertion can be tested by calculating the relative frequencies (making experimental estimates of probabilities).

- **Stochastic independence does not imply physical independence.**