Combinatorial Problems
Conditional Probability

ECE 313

Probability with Engineering Applications

Lecture 3

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Combinatorial Problems

- If the event E consists of k sample points, then

$$P(E) = \frac{\text{number of points in E}}{\text{number of points in S}} = \frac{\text{favorable outcomes}}{\text{total outcomes}} = \frac{k}{n}$$

- <u>Ordered samples of size k, with replacement</u> *(permutations with replacement) P(n, k)*

  gives the number of ways we can select k objects among n objects where order is important and when the same object is allowed to be repeated any number of times; the required number is $\boldsymbol{n^k}$

- <u>Example:</u> Find the probability that some randomly chosen k-digit decimal number is a valid k-digit octal number.

  The sample space is $S = \{(x_1, x_2, \ldots x_k) \mid x_1, \in \{0, 1, 2, \ldots, 9\}\}$

  The events of interests is $E = \{(x_1, x_2, \ldots x_k) \mid x_1, \in \{0, 1, 2, \ldots, 7\}\}$

  $|S| = 10^k$ and $|E| = 8^k$ ----> $P(E) = |E| / |S| = 8^k / 10^k$

Combinatorial Problems (cont.)

- Ordered Samples of size k, without replacement *(permutations without replacement)*
- Counts the number of ordered sequences without repetition of the same element(s); the number is given by:

$$n(n-1)....(n-k+1) = \frac{n!}{(n-k)!} \qquad k = 1, 2, \ldots, n$$

- Example: Find the probability that a randomly chosen three-letter sequence will not have any repeat letters.

Let I = {a, b, …, z} be the alphabet of 26 letters

$S = \{(\alpha, \beta, \gamma) \mid \alpha \in I, \beta \in I, \gamma \in I\}$

$E = \{(\alpha, \beta, \gamma) \mid \alpha \in I, \beta \in I, \gamma \in I, \alpha \neq \beta, \beta \neq \gamma, \alpha \neq \gamma\}$

$|E| = P(26, 3) = 15{,}600$; $\qquad |S| = 26^3 = 17{,}576$

$P(E) = 15{,}600 / 17{,}576 = 0.8875739$

Combinatorial Problems (cont.)

- Unordered sample of size k, without replacement *(combinations)* gives the number of unordered sets of distinct elements; the number is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

- Example: If a box contains 75 good IC chips and 25 defective chips, and 12 chips are selected at random, find the probability that at least one chip is defective.
- The event of interest is E = “At least one chip is defective”; we use a complementary event E’ = “No chip is defective”

$|E'| = \binom{75}{12}$ $|S| = \binom{100}{12}$

P(E’) = |E’| / |S| = (75! * 88!) / (63! * 100!) and P(E) = 1 - P(E’)

Binomial Theorem

- The values $\binom{n}{k}$ are called *binomial coefficients*

Binomial Theorem:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Combinatorial Proof of the Binomial Theorem:

- Consider the product $(x1 + y1)(x2 + y2)\cdots(xn + yn)$
- Its expansion consists of the sum of 2*n* terms, each term being the product of *n* factors. Furthermore, each of the 2*n* terms in the sum will contain as a factor either *xi* or *yi* for each *i* = 1,2,...,*n*.
- Now, how many of the 2*n* terms in the sum will have *k* of the *xi*'s and (*n*−*k*) of the *yi*'s as factors?
- As each term consisting of *k* of the *x* 's and (*n* − *k*) of the *y* 's corresponds to a choice of a group of *k* from the *n* values *x*1,*x*2,...,*xn*, there are $\binom{n}{k}$ such terms.

Example 1

- Suppose we have 100 different Xbox game CDs at the Grainger library. In each of the following cases, what are the number of possible outcomes?

a. We ask 3 different students about their favorite Xbox game.

b. We ask one student to give a ranking of the best 3 games that the student has played so far.

c. We ask one student to pick 3 of her/his favorite games to rent out.

Example 1
Solution

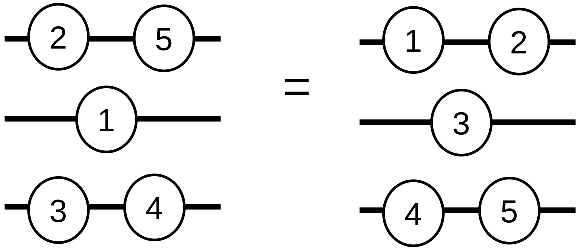
a. Permutations with replacement:
   - $(100)^3 = 1,000,000$

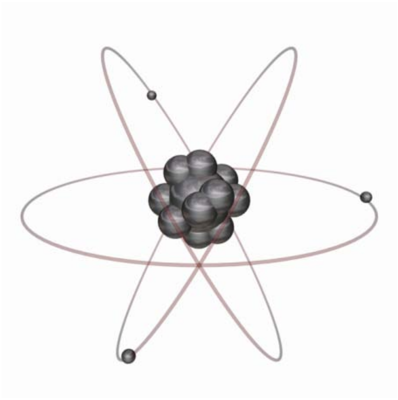
b. Permutations without replacement:
   - $100 \times 99 \times 98 = 100! / (100-3)! = 100! / 97! = 970,200$

c. Combinations:
   - $\binom{100}{3} = 100! / (3!\ 97!) = 161,700$

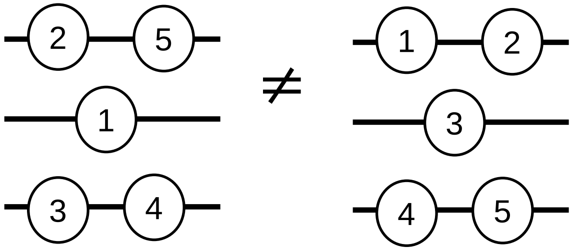
Example 2

- Assume that you have five electrons and three orbits. In how many ways can you distribute the electrons into orbits so that no orbit contains more than two electrons? (The ordering of electrons on each orbit is not important).
  - Assuming that the electrons are identical:





  - Assuming that the electrons are distinguishable (not identical):



Example 2
Solution

- Assuming that the electrons are identical:
  - There are **3 different ways** to assign 5 identical electrons to 3 different orbits. We can show it as a sequence of number of electrons assigned to each orbit:

    (Orbit 1, Orbit 2, Orbit 3) = (1, 2, 2) or (2, 1, 2) or (2, 2, 1)

- Assuming that the electrons are distinguishable (not identical):
  - If the electrons are distinguishable, there are 3 different ways to assign the 5 electrons to 3 orbits and in each case we have 30 ways to select the electrons, so in total there are **90 different ways**:
    - For case (1,2,2): $\binom{5}{1}\binom{4}{2}\binom{2}{2} = 30$
    - For case (2,1,2): $\binom{5}{2}\binom{3}{1}\binom{2}{2} = 30$
    - For case (2,2,1): $\binom{5}{2}\binom{3}{2}\binom{1}{1} = 30$

Example 3

- A bag contains n pairs of shoes in distinct styles and sizes. You pick two shoes at random from the bag. Note that this is sampling without replacement.
  a) What is the probability that you get a pair of matching shoes?
  b) What is the probability of getting one left shoe and one right shoe?

- Suppose now that $n \geq 2$ and that you choose 3 shoes at random from the bag.
  c) What is the probability that you have a pair of matching shoes among the three that you have picked?
  d) What is the probability that you picked at least one left shoe and at least one right shoe?

Example 3
Solution

a. Consider pairs of shoes as being unordered, that is, it does not matter which shoe of the pair is picked first. There are $\binom{2n}{2}$ pairs in total and among these n are matching thus the probability is:

$$\frac{n}{\binom{2n}{2}} = \frac{n}{2n(2n-1)/2} = \frac{1}{2n-1}$$

b. To get one left shoe and one right shoe, we need the second shoe to be of the opposite type of the first shoe. After picking the first shoe, among the 2n − 1 remaining shoes, n are of the opposite type. Thus, the probability of getting one left and one right shoe is n/ (2n − 1).

There are other ways to solve this problem. Try them!

Example 3
Solution

c. There are *n(2n − 2)* unordered triples including a matching pair, among the $\binom{2n}{3}$ possible triples. Thus, the answer is:

$$\frac{n(2n-2)}{\binom{2n}{3}} = \frac{n(2n-2)}{2n(2n-1)(2n-2)/6} = \frac{3}{2n-1}$$

c. There are $2.\binom{n}{3}$ all-left or all-right triples. Thus the probability of a mixed triple is:

$$1-\frac{2\binom{n}{3}}{\binom{2n}{3}} = 1-\frac{n(n-1)(n-2)/3}{2n(2n-1)(2n-2)/6} = 1-\frac{(n-2)}{2(2n-1)} = \frac{3n}{4n-2}$$

Conditional Probability

- Probability of an Event A assuming that the outcome *s* is in a subset B (or Event B).
- Called conditional probability of Event A given that Event B has occurred.

  Conditional probability A given B: (P(A/B))
- Given B has occurred, the sample point (element) corresponding to this conditional probability must be in B and not in $\overline{B}$.
- <u>Define</u>

$$P(s/B) = \begin{cases} \dfrac{P(s)}{P(B)} & \text{if } s \in B \\ 0 & \text{if } s \in \overline{B} \end{cases}$$

(original probability of a sample point is scaled by $1/P(B)$ so that the probability of all sample points in B add up to 1)

Conditional Probability (cont.)

$$P(A/B) = \sum_{s \in A} \mathrm{P(s/B)}$$

noting that

$$A = (A \cap B) \cup (A \cap \overline{B})$$

$$P(A/B) = \sum_{s \in A \cap \overline{B}} P(s/B) + \sum_{s \in A \cap B} P(s/B)$$

$$= \sum_{s \in A \cap B} P(s)/P(B)$$

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

Conditional Probability

- *Conditional probability of A given B (P(A|B))* defines the conditional probability of the event A given that the event B occurs and is given by: $P(A|B)=\dfrac{P(A\cap B)}{P(B)}$

  if $P(B)\neq 0$ and is undefined otherwise.
- A rearrangement of the above definition gives the following *multiplication rule (MR)*

$$P(A\cap B)=\begin{cases} P(B)P(A|B) & \text{if } P(B)\neq 0 \\ P(A)P(B|A) & \text{if } P(A)\neq 0 \\ 0 & \text{otherwise} \end{cases}$$

Example

- Remember Task 1 of Mini Project 1:
- There are 275,790 files submitted over the period of 3 months
- Suppose in your data set you have 5,000 failure entries, including:
  - 1,200 User Data Failures
  - 3,800 Platform Failures

- What is the probability of a User Data Failure? *(Unconditional Probability)*
  - P(User Data Failure) = 1,200/275,790 = 0.4%

- If a failure happens, what is the probability of that failure being a User Data Failure? *(Conditional Probability)*
  - P(User Data Failure | Failure Occurred) = 1200/5000 = 24%