Exam Review

ECE 313

Probability with Engineering Applications

Lecture 26

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Final Project Timeline

- **Final Presentation, Saturday, Dec. 14:**
  - **Time: 13:00 – 15:30 pm – Location: CSL 301**
  - **All the groups should be there at 12:45pm.**
  - **Each group: 10 mins for presentation, 2 mins for questions**
  - **Presentation ~ 12 slides:**
    - **1 Slide: Project topic and summary**
    - **1-2 Slides: Data collection technique and a sample of data**
    - **2 Slides: Data analysis approach and concepts used from the class**
    - **5 Slides: Results of analysis**
      - Provide evidence that your data and results are consistent
    - **1-2 Slides: Summary, conclusions, key insights, suggestions**
      - Your insights on the results are important.
      - Provide two constructive suggestions on how to improve projects

- **Final Project Report, December 19, 5:00pm.**

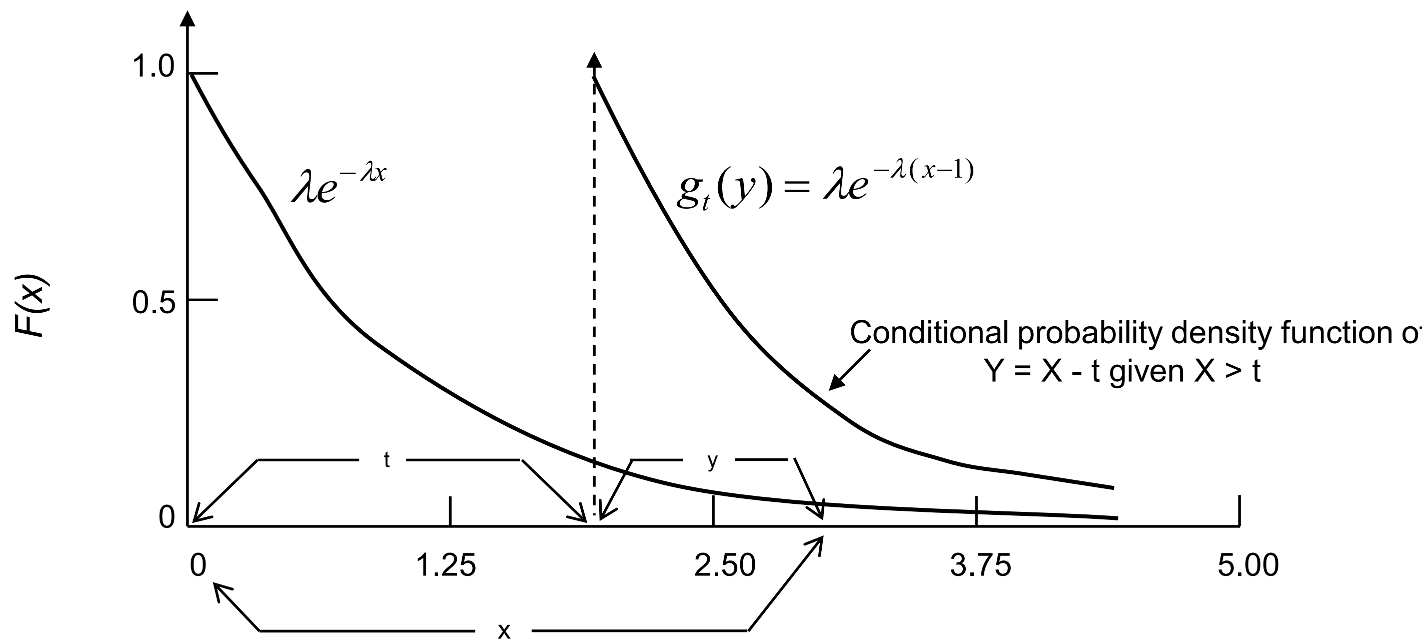
Final Exam Important Dates

- **Final Exam Review Session 1, Friday, Dec. 13,**
  - **Time: 3:00 – 4:15pm – Location: CSL 141**
- **Final Exam Review Session 2, Monday, Dec. 16,**
  - **Time: 12:00 – 1:30pm – Location: CSL 141**
  - **Time: 4:30 – 6:00pm – Location: CSL 301**

- **Final Exam, Wednesday, Dec 18,**
  - **Time: 1:30 – 4:30pm – Location 1SIEBL-1302**
  - You are allowed to bring three 8”x11” sheet of notes.
  - No calculators, laptops, or electronic gadgets are allowed.
  - Expected to be between 2.5 – 3 hours, approx. 6-8 problems.
  - Approx. 60% from the material after the midterm exam, and 40% from the material before the midterm, or a mixture of both
  - Show all your work to get partial credit.

Exam Review: Memory-less Property

- **Memory-less property of Exponential**



Review: Expectations

- **Expectation:**
  - The Discrete Case $E[X] = \sum_{x:p(x)>0} xp(x)$
  - The Continuous Case $E[X] = \int_{-\infty}^{\infty} xf(x)dx$

- **Expectation of function of a random variable**

$$E[g(X)] = \sum_{x:p(x)>0} g(x)p(x)$$

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)dx$$

  - **Corollary:**

$$E[aX + b] = aE[X] + b$$

Review: Moments

- **Moments:**

$$E[Y]=E[\phi(X)]=\begin{cases}\sum_i \phi(x_i)p_X(x_i), & if\ X\ is\ discrete,\\ \int_{-\infty}^{\infty}\phi(x)f_X(x)dx, & if\ X\ is\ continuous,\end{cases}$$

$$Y=X^n => E[X^n],\ n\geq 1 \qquad \mu_k=E[(X-E[X])^k]$$

- **Variance:**

$$Var[X]=\mu^2=\sigma^2 x=\begin{cases}\sum_i (x_i-E[X])^2 p(x_i) & if\ X\ is\ discrete\\ \int_{-\infty}^{\infty}(x-E[X])^2 f(x)dx & if\ X\ is\ continuous\end{cases}$$

$$Var(X)=E[X^2]-(E[X])^2$$

- **Corollary:**

$$Var[aX+b]=a^2 Var(X)$$

Review: Functions of Random Variables

- **Functions of random variables**
  - Examples shown in the class

- **Reliability function and Mean time to failure:**
  - Let *T* denote the time to failure or lifetime of a component in the system

$$R(t) = P\{T > t\} = 1 - F(t)$$

$$E[T] = \int_0^\infty R(t)\,dt = MTTF$$

  - If the component lifetime is exponentially distributed, then:

$$R(t) = e^{-\lambda t}$$

$$E[T] = \int_0^\infty e^{-\lambda t}\,dt = \frac{1}{\lambda}$$

$$Var[T] = \int_0^\infty 2te^{-\lambda t}\,dt - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

Review: Joint Distribution Functions

- **Joint distribution functions:**
  - For any two random variables X and Y, the ***joint cumulative probability distribution function*** of X and Y by:

$$F(a,b) = P\{X \le a, Y \le b\}, \ -\infty < a, b < \infty$$

  - **Discrete:**
    - The ***joint probability mass function*** of X and Y

$$p(x, y) = P\{X = x, Y = y\}$$

    - Marginal PMFs of X and Y:

$$p_X(x) = \sum_{y: p(x,y)>0} p(x, y)$$

$$p_Y(y) = \sum_{x: p(x,y)>0} p(x, y)$$

Review: Joint Distribution Functions

- **Joint distribution functions:**
  - **Continuous:**
    - The ***joint probability density function*** of X and Y:

$$P\{X \in A, Y \in B\} = \int_B \int_A f(x, y)\,dx\,dy$$

    - Marginal PDFs of X and Y:

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)\,dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y)\,dx$$

    - Relation between joint CDF and PDF

$$F(a,b) = P(X \le a, Y \le b) = \int_{-\infty}^{a} \int_{-\infty}^{b} f(x, y)\,dy\,dx$$

Review: Joint Distribution Functions

- Function of two joint random variables

$$E[g(X,Y)] = \sum_{y}\sum_{x} g(x,y)\,p(x,y)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\,f(x,y)\,dx\,dy$$

- For example, if $g(X,Y)=X+Y$, then, in the continuous case

$$E[X+Y] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x+y)\,f(x,y)\,dx\,dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xf(x,y)\,dx\,dy + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} yf(x,y)\,dx\,dy$$

$$= E[X] + E[Y]$$

Independent Random Variables

- **Independent Random Variables:** Two random variables X and Y are said to be independent if:

$$F(x, y) = F_X(x)F_Y(y), \ -\infty < x < \infty, -\infty < y < \infty$$

- If X and Y are continuous:

$$f(x, y) = f_X(x)f_Y(y), \ -\infty < x < \infty, -\infty < y < \infty$$

- If X is discrete and Y is continuous:

$$P(X = x, Y \le y) = p_X(x)f_Y(y), \text{ all } x \text{ and } y$$

Review: Binary Hypothesis Testing

- H_0 = Negative or null hypothesis,
- H_1 = Positive or alternative hypothesis

- ***likelihood matrix***: The matrix of *conditional probabilities* of $P(X = k \mid H_i)$
- ***joint probability matrix***, The matrix of *joint probabilities* of $P(H_i, X = k)$
  - Requires *prior probabilities* of $\pi_0 = P(H_0) \quad and \quad \pi_1 = P(H_1)$

$$p_{false-alarm} = P(declare\ H_1\ true \mid H_0)$$

sum of entries in the H_0 row of the likelihood matrix that are not underlined

$$p_{miss} = P(declare\ H_0\ true \mid H_1)$$

sum of entries in the H_1 row of the likelihood matrix that are not underlined

$$p_e = \pi_0 p_{false-alarm} + \pi_1 p_{miss}$$

sum of all the entries in the *joint probability* matrix that are not underlined.

Review: Likelihood Ratio Test (LRT)

- Likelihood ratio test:

$$\Lambda(k) = \frac{p_1(k)}{p_0(k)} = \frac{P(X = k \mid H_1)}{P(X = k \mid H_0)}$$

$$\Lambda(X)\begin{cases} > \tau & declare\ H_1\ is\ true. \\ < \tau & declare\ H_0\ is\ true. \end{cases}$$

- ML Decision Rule $\tau = 1$:

$$\Lambda(X)\begin{cases} > 1 & declare\ H_1\ is\ true. \\ < 1 & declare\ H_0\ is\ true. \end{cases}$$

- MAP Decision Rule $\tau = \frac{\pi_0}{\pi_1}$:

$$\Lambda(X)\begin{cases} > \dfrac{\pi_0}{\pi_1} & declare\ H_1\ is\ true. \\ < \dfrac{\pi_0}{\pi_1} & declare\ H_0\ is\ true. \end{cases}$$

Review: Covariance and Variance

- The covariance of any two random variables, X and Y, denoted by $Cov(X,Y)$, is defined by

$$\begin{aligned} Cov(X,Y) &= E[(X-E[X])(Y-E[Y])] \\ &= E[XY - YE[X] - XE[Y] + E[X]E[Y]] \\ &= E[XY] - E[Y]E[X] - E[X]E[Y] + E[X]E[Y] \\ &= E[XY] - E[X]E[Y] \end{aligned}$$

- If X and Y are independent then it follows that $Cov(X,Y)=0$

- For any random variable X, Y, Z, and constant c, we have:
  1. Cov (X,X) = Var(X),
  2. Cov (X,Y) = Cov(Y,X),
  3. Cov (cX,Y) = cCov(X,y),
  4. Cof (X,Y+Z) = Cov(X,Y) + Cov(X,Z).

Review: Covariance and Variance

- **Covariance and Variance of Sums of Random Variables**

$$Cov\left(\sum_{i=1}^{n} X_i, \sum_{j=1}^{m} Y_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} Cov(X_i, Y_j)$$

- A useful expression for the variance of the sum of random variables can be obtained from the preceding equation

$$Cov\left(\sum_{i=1}^{n} X_i, \sum_{j=1}^{m} Y_j\right) = \sum_{i=1}^{n} Var(X_i) + 2\sum_{i=1}^{n}\sum_{j<i} Cov(X_i, Y_j)$$

- If $X_i, i = 1,\ldots,n$ are independent random variables, then the above equation reduces to

$$Var\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} Var(X_i)$$

Review: Limit Theorems

- **Markov's Inequality:** If X is a random variable that takes only nonnegative values, then for any value $a>0$:

$$P\{X \geq a\} \leq \frac{E[X]}{a}$$

- **Chebyshev's Inequality:** If X is a random variable with mean μ and variance σ^2 then for any value $k>0$,

$$P\{|X-\mu| \geq k\} \leq \frac{\sigma^2}{k^2}$$

- **Strong law of large numbers:** Let $X_1, X_2, \ldots$ be a sequence of independent random variables having a common distribution, and let $E[X_i]=\mu$ Then, with probability 1,

$$\frac{X_1 + X_2 + \ldots + X_n}{n} \to \mu \quad \text{as } n \to \infty$$

Review: Limit Theorems

- **Central Limit Theorem:** Let $X_1, X_2, \ldots$ be a sequence of independent, identically distributed random variables, each with mean μ and variance σ^2 then the distribution of $\dfrac{X_1 + X_2 + \ldots + X_n - n\mu}{\sigma\sqrt{n}}$

- Tends to the standard normal as $n \to \infty$. That is,

$$P\left\{\frac{X_1 + X_2 + \ldots + X_n - n\mu}{\sigma\sqrt{n}} \le a\right\} \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-x^2/2} dx$$

as $n \to \infty$.

- Note that like other results, this theorem holds for any distribution of the X_i 's; herein lies its power.

Review: Hazard Function

- The time to failure of a component is the random variable T. Therefore the failure density function is defined by:

$$P(t < \mathrm{T} \le t + dt) = f(t)dt$$

- The probability of failure between time t and $t+dt$, given that there were no failures up to time t:

$$P(t < \mathrm{T} \le t + dt \mid \mathrm{T} > t) = \frac{P(t < \mathrm{T} \le t + dt)}{P(\mathrm{T} > t)}$$

- Where: $P(\mathrm{T} > t) = 1 - P(\mathrm{T} < t) = 1 - F(t) = R(t)$ **=> Reliability Function**

- **Hazard Function**:

$$z(t) = \lim_{dt \to 0} \frac{P(t < \mathrm{T} \le t + dt \mid T > t)}{dt} \quad => \quad z(t) = \frac{f(t)}{1 - F(t)}$$

- An exponential failure density $f(t) = \lambda e^{-\lambda t}$ corresponds to constant hazard function λ

Review: Hazard Rate

- Instantaneous hazard rate:

$$R(t) = \exp\left[-\int_0^t h(x)dx \right]$$

- Cumulative hazard rate:

$$H(t) = \int_0^t h(x)dx$$

$$R(t) = e^{-H(t)}$$

- If the lifetime is exponentially distributed, then $H(t) = \lambda t$ and we obtain the exponential reliability function.

Constraints on f(t) and z(t)

Density Function	*Hazard Rate*
f(t) for $0 < t \leq \infty$ Density function is defined for all positive time	z(x) for $0 < t \leq \infty$ Hazard rate is defined for all positive time
$f(t) \geq 0$ f(t) is never negative	$z(t) \geq 0$ z(t) is never negative
$\int_0^\infty f(t)dt = 1$ Probability of sample space is unity	$\int_0^\infty z(t)dt = \infty$ Equivalent to condition on f(t)

Review: Treatment of Failure Data

- The probability density function over the time interval $t_i < t \le t_i + \Delta t_i$

$$f_d(t) = \frac{[n(t_i) - n(t_i + \Delta t_i)]/N}{\Delta t_i} \quad for\, t_i < t \le t_i + \Delta t_i$$

- The data hazard (inst. failure rate) over the interval $t_i < t \le t_i + \Delta t_i$:

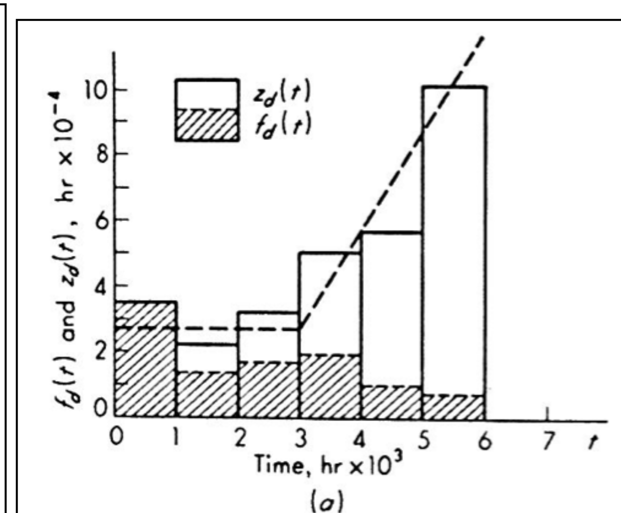
$$z_d(t) = \frac{[n(t_i) - n(t_i + \Delta t_i)]/n(t_i)}{\Delta t_i} \quad for\, t_i < t \le t_i + \Delta t_i$$

Table 4.3 Failure data for 172 hypothetical components

Time interval, hr	*Failures in the interval*
0–1,000	59
1,001–2,000	24
2,001–3,000	29
3,001–4,000	30
4,001–5,000	17
5,001–6,000	13
	Total 172

Table 4.4 Failure rates of hypothetical component

Time interval, hr	*Failure density* $f_d(t)(\times 10^{-4})$	*Hazard rate* $z_d(t)(\times 10^{-4})$
0–1,000	$\frac{59}{172 \times 10^3} = 3.43$	$\frac{59}{172 \times 10^3} = 3.43$
1,001–2,000	$\frac{24}{172 \times 10^3} = 1.40$	$\frac{24}{113 \times 10^3} = 2.12$
2,001–3,000	$\frac{29}{172 \times 10^3} = 1.69$	$\frac{29}{89 \times 10^3} = 3.26$
3,001–4,000	$\frac{30}{172 \times 10^3} = 1.74$	$\frac{30}{60 \times 10^3} = 5.00$
4,001–5,000	$\frac{17}{172 \times 10^3} = 0.99$	$\frac{17}{30 \times 10^3} = 5.69$
5,001–6,000	$\frac{13}{172 \times 10^3} = 0.76$	$\frac{13}{13 \times 10^3} = 10.00$



Review: Phase-type exponential distributions

- Exponential Distribution:
  - Time to the event or Inter-arrivals => Poisson

- **Phase-type Exponential Distributions:**
- We have a process that is divided into *k* sequential phases, in which time that the process spends in each phase is:
  - Independent
  - Exponentially distributed

– The generalization of the phase-type exponential distributions is called **Coxian Distribution**
  - Any distribution can be expressed as the sum of phase-type exponential distributions

Review: Phase-type exponential distributions

Four special types of phase-type exponential distributions:

1) Hypoexponential Distribution:

- Exponential distributions at each phase have different λ

2) K-stage Erlang Distribution:

- Exponential distributions in each phase are identical (with same λ)
- The number of phases (α) is an <u>integer</u>

3) Gamma Distribution

- Is a K-stage Erlang
- But the number of phases (α) is <u>not an integer</u>

4) Hyperexponential Distribution:

- A mixture of different exponential distributions

Review: Phase-type exponential distributions

- ***k*-phase hyperexponential**:
  - PDF: $f(t) = \sum_{i=1}^{k} \alpha_i \lambda_i e^{-\lambda_i t}, \qquad t > 0, \lambda_i > 0, \alpha_i > 0, \sum_{i=1}^{k} \alpha_i = 1$
  - CDF: $F(t) = \sum_{i} \alpha_i (1 - e^{-\lambda_i t}), \qquad \mathrm{t} \geq 0$
  - Failure rate: $h(t) = \dfrac{\sum \alpha_i \lambda_i e^{-\lambda_i t}}{\sum \alpha_i e^{-\lambda_i t}}, \qquad \mathrm{t} > 0$

Covariance and Variance Review

- The covariance of any two random variables, X and Y, denoted by $Cov(X,Y)$, is defined by

$$
\begin{aligned}
Cov(X,Y) &= E[(X - E[X])(Y - E[Y])] \\
&= E[XY - YE[X] - XE[Y] + E[X]E[Y]] \\
&= E[XY] - E[Y]E[X] - E[X]E[Y] + E[X]E[Y] \\
&= E[XY] - E[X]E[Y]
\end{aligned}
$$

- If X and Y are independent then it follows that $Cov(X,Y)=0$

- For any random variable X, Y, Z, and constant c, we have:
  1. Cov (X,X) = Var(X),
  2. Cov (X,Y) = Cov(Y,X),
  3. Cov (cX,Y) = cCov(X,Y),
  4. Cov (X,Y+Z) = Cov(X,Y) + Cov(X,Z).

Covariance and Variance Review

- **Covariance and Variance of Sums of Random Variables**

$$Cov\left(\sum_{i=1}^{n} X_i, \sum_{j=1}^{m} Y_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} Cov(X_i, Y_j)$$

- A useful expression for the variance of the sum of random variables can be obtained from the preceding equation

$$Cov\left(\sum_{i=1}^{n} X_i, \sum_{j=1}^{m} Y_j\right) = \sum_{i=1}^{n} Var(X_i) + 2\sum_{i=1}^{n}\sum_{j<i} Cov(X_i, Y_j)$$

- If $X_i, i = 1,\ldots,n$ are independent random variables, then the above equation reduces to

$$Var\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} Var(X_i)$$

Cold Standby Example

- Consider a dual redundant system composed of two identical components, where the secondary component acts as a backup of the primary one and will be powered on only after the primary component fails.
- A detector circuit checks the output of primary component in order to identify it failure and a switch is used to configure and power on the secondary component.
- Let the lifetime of the two components be two independent random variables $X1$ and $X2$, which are exponentially distributed with parameter λ.

- Assume that the detection and switching circuits are perfect. What is the distribution of time to failure of the whole system? Derive the reliability function for the system.

Cold Standby Example (Cont'd)

- Total lifetime of the system can be modeled by random variable Z which is a **2-stage Erlang distribution**.

- Remember for an r-phase Erlang distribution we have:

$$f(t) = \frac{\lambda^r t^{r-1} e^{-\lambda t}}{(r-1)!}, \quad t > 0, \lambda > 0, r = 1,2,...$$

$$F(t) = 1 - \sum_{k=0}^{r-1} \frac{(\lambda t)^k}{k!} e^{-\lambda t}, \quad t \le 0, \lambda > 0, r = 1,2,...$$

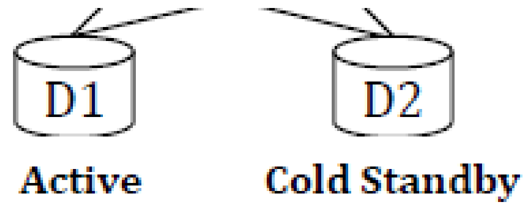
- So for 2-stage Erlang we have:

$$F_z(t) = \int_0^t f_z(z)\,dz = 1 - (1 + \lambda t)e^{-\lambda t}$$

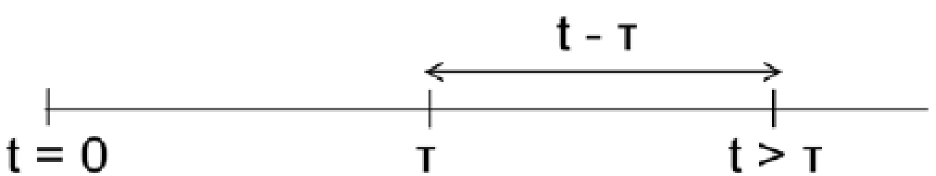
$$R(t) = 1 - F(t) = F_z(t) = (1 + \lambda t)e^{-\lambda t}, t \ge 0$$

Cold Standby Example (Cont'd)

- Remember the example in Class Project 3



If we assume that the first disk array fails at some time τ, then the lifetime of second disk array starts at time τ, and it fails at a time t > τ. So the lifetime of the second disk will be t – τ:



The probability density function for the failure of the first disk array is:

$$f_1(t) = \lambda_a e^{-\lambda_a t}, 0 < \tau < t$$

Given that the first disk array must fail for the lifetime of the second disk array to start, the density function of the lifetime of the second disk array is conditional, given by:

$$f_2(t \mid \tau) = \begin{cases} \lambda_a e^{-\lambda a(t-\tau)}, & 0 < \tau < t \\ 0, & \tau > t \end{cases}$$

Cold Standby Example (Cont'd)

Then we define the system failure as a function of t and τ, using the definition of conditional probability: $\varphi(t,\tau) = f_1(\tau) f_2(t \mid \tau)$

The marginal density function of *f(t)* (failure of system composed of 2-disk arrays) is:

$$f(t) = \int_0^t \varphi(t,\tau)\, d\tau = \int_0^t \lambda_a{}^2 e^{-\lambda_a t} d\tau = \lambda_a{}^2 e^{-\lambda_a t}.\, [1]_0^t = \lambda_a{}^2 t e^{-\lambda_a t}$$

So the reliability function *R(t)* for the two disk array system will be calculated as follows:

$$R_{2disks}(t) = 1 - \int_0^t f(t) dt = \int_0^t \lambda_a{}^2 t e^{-\lambda_a t} dt$$

Integrating by parts, we get:

$$R_{2disks}(t) = (1 + \lambda_a t) e^{-\lambda_a t}$$

- This is a two-stage Erlang distribution with the parameter λ_a

Cold Standby Example (Cont'd)

- Now, assume that the lifetime of the two components (X1 and X2), are exponentially distributed with different parameters: $\lambda 1$ and $\lambda 2$.

- Total lifetime of the system can be modeled by random variable Z, which is a **2-stage Hypo-exponential distribution**.
- Remember that a two-stage hypoexponential random variable, *X*, with parameters λ_1 and λ_2 ($\lambda_1 \neq \lambda_2$), is denoted by *X*~HYPO(λ_1, λ_2).

$$f(t) = \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1}(e^{-\lambda_1 t} - e^{-\lambda_2 t}), \qquad t > 0$$

- So the distribution of time to failure of the system would be:

$$F(t) = 1 - \frac{\lambda_2}{\lambda_2 - \lambda_1} e^{-\lambda_1 t} + \frac{\lambda_1}{\lambda_2 - \lambda_1} e^{-\lambda_2 t}, \qquad t \geq 0$$