Exam Review

ECE 313

Probability with Engineering Applications

Lecture 25

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Final Project Timeline

- **Project Summary: Saturday, Nov. 16, 5:00pm**
  - Submit a one-paragraph summary of your goals and overall plans
- **Short Progress Reports: Every Tuesday, starting Nov. 19.**
  - A short paragraph (3-4 lines) on what progress you made and what is the plan for the next week

- **First Presentation, this Thursday, Nov. 21, in the class**
  - 2 slides showing your overall plan, timeline, the division of tasks, and techniques are your going to use.
- **Intermediate Presentation, Tuesday, Dec. 3, in the class**
  - Reporting on your progress

- **Final Presentation, Saturday, Dec. 14, in the class**
- **Final Exam Review Session, Monday, Dec. 16, Time: TBD**
- **Final Project Report, December 19, 5:00pm.**

Review: Basic Concepts

- Basic Concepts:
  - ***Random experiment*** is an experiment the outcome of which is not certain
  - ***Sample Space (S)*** is the totality of the possible outcomes of a random experiment
  - ***Discrete (countable) sample space*** is a sample space which is either
    - *finite*, i.e., the set of all possible outcomes of the experiment is finite
    - *countably infinite, i.e.,* the set of all outcomes can be put into a one-to-one correspondence with the natural numbers
  - ***Continuous sample space*** is a sample space for which all elements constitute a continuum, such as all the points on a line, all the points in a plane
  - An ***event*** is a collection of certain sample points, i.e., a subset of the sample space
    - ***Universal event*** is the entire sample space S
    - *The null set* $\varnothing$ is a **null or impossible event**

Review: Algebra of Events

- **Algebra of Events**
  - The ***intersection*** of E_1 and E_2 is given by:
    - $E_1 \cap E_2$ = {s $\in$ S |s is an element of both E_1 and E_2}
  - The ***union*** E_1 and E_2 is given by:
    - $E_1 \cup E_2$ = = {s $\in$ S |either s $\in E_1$ or s $\in E_2$ or both}
  - In general: $|E_1 \cup E_2| \leq |E_1| + |E_2|$
    - where |A| = the number of elements in the set **(Cardinality)**
  - Definition of *union* and *intersection* extend to any finite number of sets:

$$\bigcup_{i=1}^{n} E_i = E_1 \cup E_2 \cup E_3 \cup \ldots \cup E_n$$

$$\bigcap_{i=1}^{n} E_i = E_1 \cap E_2 \cap E_3 \cap \ldots \cap E_n$$

Review: Mutually Exclusive Events

- ***Mutually exclusive or disjoint events*** are two events for which
$$A \cap B = \varnothing$$

- A list of events $A_1, A_2, \ldots, A_n$ is said to be
  - composed of ***mutually exclusive events*** iff:
$$A_i \cap A_j = \begin{cases} A_i & \text{if } i = j \\ \varnothing & \text{otherwise} \end{cases}$$
  - *collectively* ***exhaustive*** iff: $A_1 \cup A_2 \cup \ldots \cup A_n = S$

Review: Probability Axioms

- **Probability Axioms**
  - Let *S* be a sample space of a random experiment and *P(A)* be the probability of the event *A*
  - The probability function *P(.)* must satisfy the three following axioms:
  - **(A1)** For any event A, P(A) ≥ 0

    (probabilities are nonnegative real numbers)
  - **(A2)** P(S) = 1

    (probability of a certain event, an event that must happen is equal 1)
  - **(A3)** $P(A \cup B) = P(A) + P(B)$, whenever A and B are mutually exclusive events, i.e., $A \cap B = \varnothing$

    (probability function must be additive)
  - **(A3')** For any countable sequence of events $A_1, A_2, \ldots, A_n \ldots$, that are mutually exclusive (that is $A_j \cap A_k = \varnothing$ whenever $j \neq k$)

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n)$$

Review: Probability Axioms

- **(Ra)** For any event A, $P(\overline{A}) = 1 - P(A)$
- **(Rb)** If $\varnothing$ is the impossible event, then $P(\varnothing) = 0$
- **(Rc)** If A and B are any events, not necessarily mutually exclusive, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- **(Rd)**(generalization of Rc) If $A_1, A_2, \ldots, A_n$ are any events, then

$$P\left(\bigcup_{i=1}^{n} A_i\right) = P(A_1 \cup A_2 \cup \ldots \cup A_n) = \sum_i P(A_i) - \sum_{1 \le i < j \le n} P(A_i \cap A_j)$$

$$+ \sum_{1 \le i < j < k \le n} P(A_i \cap A_j \cap A_k) + \ldots + (-1)^{n-1} P(A_1 \cap A_2 \cap \ldots \cap A_n)$$

where the successive sums are over all possible events, pairs of events, triples of events, and so on.

(Can prove this relation by induction (see class web site))

Review: Combinatorial Problems

- **Combinatorial Problems**
  - Permutations with replacement:
    - Ordered samples of size k, with replacement *P(n, k)*
  - Permutations without replacement
    - Ordered Samples of size k, without replacement

$$n(n-1)....(n-k+1) = \frac{n!}{(n-k)!} \qquad k = 1, 2, \ldots, n$$

  - Combinations
    - Unordered sample of size k, without replacement

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

- **Binomial Theorem**

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Review: Conditional Probability

- **Conditional Probability** *of A given B (P(A|B))* defines the conditional probability of the event A given that the event B occurs and is given by:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

  if P(B) $\neq$ 0 and is undefined otherwise.

- A rearrangement of the above definition gives the following ***multiplication rule (MR)***

$$P(A \cap B) = \begin{cases} P(B)P(A \mid B) & \text{if } P(B) \neq 0 \\ P(A)P(B \mid A) & \text{if } P(A) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

- Or:

$$P(A \cap B) = P(A)P(B|A) = P(B)P(A|B)$$

Review: Bayes Formula

- **Theorem of Total Probability**
- Any event A can be partitioned into two disjoint subsets:

$$A = (A \cap B) \cup (A \cap \overline{B})$$

- Then:

$$P(A) = P(A \cap B) \cup P(A \cap \overline{B})$$

$$= P(A \mid B)P(B) + P(A \mid \overline{B})P(\overline{B})$$

- In general:

$$\boxed{P(A) = \sum_{i=1}^{n} P(A \mid B_i)P(B_i)}$$

- **Bayes Formula:**

$$\boxed{P(B_j \mid A) = \frac{P(B_j \cap A)}{P(A)} = \frac{P(A \mid B_j)P(B_j)}{\sum_i P(A \mid B_i)P(B_i)}}$$

Review: Independence of Events

- **Independence of Events:**
- Two events A and B are independent if and only if:

$$P(A|B)=P(A)$$

- Or events A and B are said to be independent if:

$$P(A \bigcap B) = P(A)P(B)$$

Review: Series and Parallel Systems

- **Reliability Applications:**
  - Recovery blocks
  - Series and parallel systems:
    - Series System: $R_s = P$ ("The system is functioning properly.")

$$= P(A_1 \cap A_2 \cdots \cap A_n)$$

$$= P(A_1)P(A_2)\cdots P(A_n)$$

$$= \prod_{i=1}^{n} R_i \qquad (2.1)$$

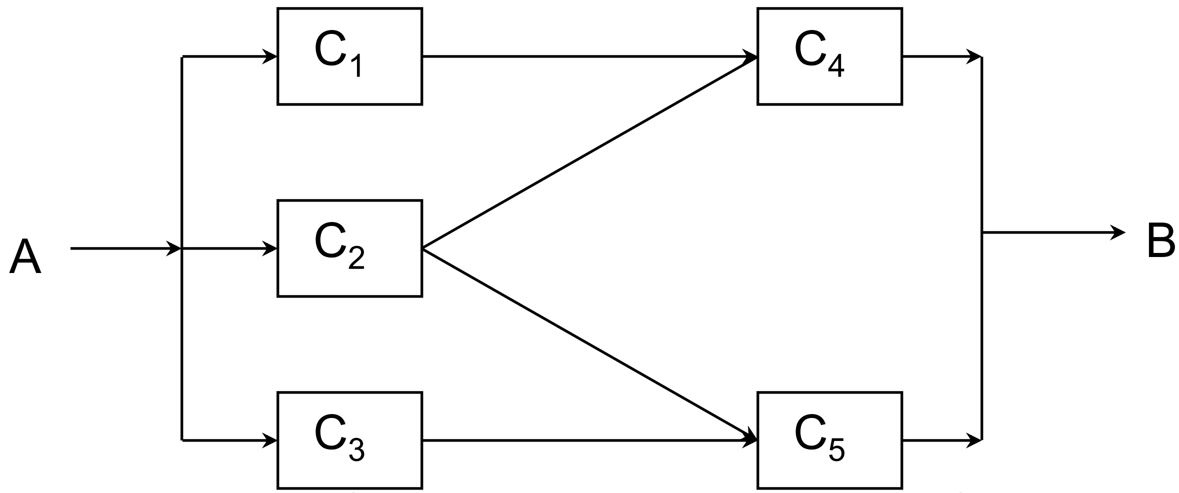
    - Parallel System:

$$R_p = 1 - F_p = 1 - \prod_{i=1}^{n} (1 - R_i)$$

    - In general: $R_{sp} = \prod_{i=1}^{n} [1 - (1 - R_i)^{n_i}]$

Review: Complex Series-Parallel Systems

- Non series-parallel systems:



  - Using the theorem of total probability and the Bayes formula:

$$P(X) = P(X \mid X_2)P(X_2) + P(X \mid \overline{X}_2)P(\overline{X}_2) = P(X \mid X_2)R_2 + P(X \mid \overline{X}_2)(1 - R_2)$$

Review: TMR System

- **Bernoulli Trials**
  - The probability of obtaining exactly k successes in n trials is :

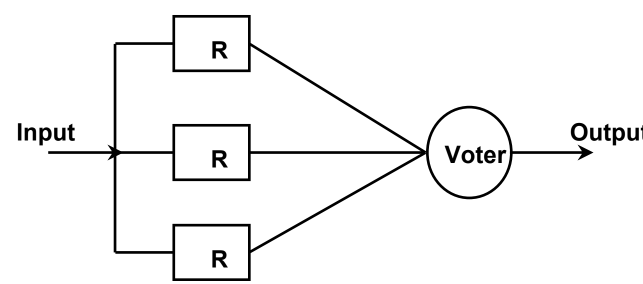
$$p(k) = \binom{n}{k} p^k q^{n-k} \qquad k = 0, 1, \ldots, n$$

- **NMR System:**

$$R_{m|n} = P(\text{"m or more components functioning properly"})$$
$$= P\left(\bigcup_{i=m}^{n} \{\text{"exactly i components functioning properly"}\}\right)$$
$$= \sum_{i=m}^{n} P(\text{"exactly i components functioning properly"})$$
$$= \sum_{i=m}^{n} p(i) = \sum_{i=m}^{n} \binom{n}{i} R^i (1-R)^{n-i}$$

- **TMR System:**

$$R_{TMR} = 3R^2 - 2R^3$$



Review: Random Variables

- **Random Variables:**
  - A random variable X on a sample space S is a function $X: S \to \Re$ that assigns a real number X(s) to each sample point $s \in S$.
  - ***Discrete*** random variables: The random variables which are either finite or countable.
    - Bernoulli
    - Binomial
    - Poisson
    - Geometric
    - Modified Geometric
  - ***Continuous*** random variables: The random variables that take on a continuum of possible values.
    - Uniform
    - Normal
    - Exponential

Review: Cumulative distribution function (cdf)

- ***Cumulative distribution function*** **(cdf) (or *distribution function*)** $F(\cdot)$ of a random variable X is defined for any real number $b, -\infty < b < \infty$, by $F(b) = P\{X \leq b\}$
- $F(b)$ denotes the probability that the random variable X takes on a value that is less than or equal to b.

- Some properties of cdf F are:
  i. $F(b)$ is a non-decreasing function of b,
  ii. $\lim_{b \to +\infty} F(b) = F(\infty) = 1,$
  iii. $\lim_{b \to -\infty} F(b) = F(-\infty) = 0.$

- All probability questions about X can be answered in terms of cdf $F(\cdot)$. e.g.: $P\{a \leq X \leq b\} = F(b) - F(a) \quad \text{for all } a < b$

Review: Discrete Random Variables

- **Discrete Random Variables:**
  - ***Probability mass function (pmf)***:

$$p(a) = P\{X = a\}$$

    - Properties:

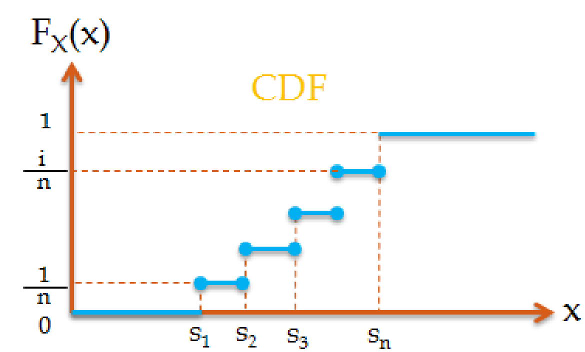
$$\begin{cases} p(x_i) > 0, & i = 1,2,\ldots \\ p(x) = 0, & \text{for other values of x} \end{cases}$$

$$\sum_{i=1}^{\infty} p(x_i) = 1$$

  - **Cumulative distribution function (CDF)**:

$$F(a) = \sum_{all\ x_i \le a} p(x_i)$$

    - **A stair step function**



Review: Continuous Random Variables

- **Continuous Random Variables:**
  - ***Probability distribution function (pdf)***:

$$P\{X \in B\} = \int_B f(x)\,dx$$

    - Properties:

$$1 = P\{X \in (-\infty, \infty)\} = \int_{-\infty}^{\infty} f(x)\,dx$$

    - All probability statements about X can be answered

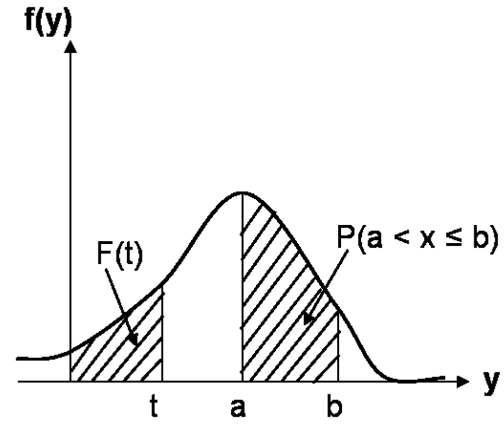
$$P\{a \le X \le b\} = \int_a^b f(x)\,dx$$

$$P\{X = a\} = \int_a^a f(x)\,dx = 0$$

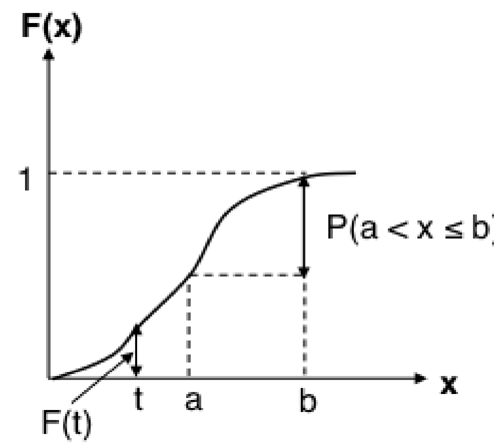
  - **Cumulative distribution function (CDF)**:

$$F_x(x) = P(X \le x) = \int_{-\infty}^{x} f_x(t)\,dt \quad , -\infty < x < \infty$$

    - Properties: $\frac{d}{da}F(a) = f(a)$
    - **A continuous function**



pdf



CDF

Review: Important Distributions

- **Summary of important distributions:**

Distribution	PDF or PMF	Mean	Variance
$Bernoulli(p)$	$\begin{cases} p, & \text{if } x = 1 \\ 1-p, & \text{if } x = 0. \end{cases}$	p	$p(1-p)$
$Binomial(n,p)$	$\binom{n}{k} p^k(1-p)^{n-k}$ for $0 \leq k \leq n$	np	npq
$Geometric(p)$	$p(1-p)^{k-1}$ for $k = 1, 2, \ldots$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
$Poisson(\lambda)$	$e^{-\lambda}\lambda^x/x!$ for $k = 1, 2, \ldots$	λ	λ
$Uniform(a,b)$	$\frac{1}{b-a}$ $\forall x \in (a,b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
$Gaussian(\mu,\sigma^2)$	$\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	μ	σ^2
$Exponential(\lambda)$	$\lambda e^{-\lambda x}$ $x \geq 0, \lambda > 0$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$