Quiz 1 and Final Project, Hyperexponential Distributions

ECE 313

Probability with Engineering Applications

Lecture 22

Professor Ravi K. Iyer

Dept. of Electrical and Computer Engineering

University of Illinois at Urbana Champaign

Today's Topics

- **Quiz 1**
- **Final Project**
- **Review:**
  - **Hypo**exponential Distribution
  - Erlang and Gamma Distributions
- **Hyper**exponential Distribution

Final Project

- Pick an example system that you want to measure and characterize its reliability or performance. For example:
  - Software-as-a-Service (SaaS) Business System (Mini Project 1)
  - Patient Monitoring System (Mini Project 2)
  - Blue Waters Supercomputers
  - Your own PC or engineering workstation network (EWS)

- Think and identify what are the uncertainties in your system behavior.
- How would you go about characterizing these uncertainties?
- What experiments would you run, how often, what background conditions would you assume, what parameters (identify at least two) would you measure?

Final Project Topics

- Come up with ideas on how to use the concepts learned in the class to study the performance or reliability of the system.
- Example topics include:
  - Reliability/Performance evaluation
  - Random variables
  - Joint and conditional distributions
  - Special distributions:
    - Binomial, Geometric, Poisson, Gaussian, Exponential, etc.
  - Measuring mean, variance, co-variance, and correlation
  - Hypothesis testing
  - Mean time to failure, hazard rates, and failure densities

- You **must** do real or simulated measurements of the system.

Final Project Deadlines

- **Individual meetings with each group, Thursday, Nov. 14**
  - **In CSL 249, starting 12:30.**
  - Write your name in one of the slots in the sign-up sheet
  - Come with some preliminary ideas on your project topic

- **First presentation (2 slides), next Thursday, Nov. 21:**
  - **Project goal and preliminary plan**
    - System under study
    - How to measure data
    - Techniques or concepts learned to apply
  - **Tasks for each group member**
  - **Timeline to finish the project**
- **Final project (report), due on the final exam date**
- **Final presentation, due the last day of the class**

Hyperexponential Distribution

- A process with sequential phases gives rise to a hypoexponential or an Erlang distribution, depending upon whether or not the phases have identical distributions.
- If a process consists of alternate phases, i. e. during any single experiment the process experiences one and only one of the many alternate phases, **and**
- If these phases have independent exponential distributions, **then**
- The overall distribution is hyperexponential.

Hyperexponential Distribution (cont.)

- The density function of a *k*-phase hyperexponential random variable is:

$$f(t) = \sum_{i=1}^{k} \alpha_i \lambda_i e^{-\lambda_i t}, \qquad t > 0, \lambda_i > 0, \alpha_i > 0, \sum_{i=1}^{k} \alpha_i = 1$$

- The distribution function is:

$$F(t) = \sum_i \alpha_i (1 - e^{-\lambda_i t}), \qquad \mathrm{t} \geq 0$$

- The failure rate is:

$$h(t) = \frac{\sum \alpha_i \lambda_i e^{-\lambda_i t}}{\sum \alpha_i e^{-\lambda_i t}}, \qquad \mathrm{t} > 0$$

which is a decreasing failure rate from $\Sigma\alpha_i\lambda_i$ down to min $\{\lambda_1, \lambda_2, \ldots\}$

Hyperexponential Distribution (cont.)

- The hyperexponential is a special case of mixture distributions that often arise in practice:

$$F(x) = \sum_i \alpha_i F_i(x), \qquad \sum \alpha_1 = 1, \alpha_1 \geq 0$$

- The hyperexponential distribution exhibits more variability than the exponential, e.g. CPU service-time distribution in a computer system often expresses this.
- If a product is manufactured in several parallel assembly lines and the outputs are merged, then the failure density of the overall product is likely to be hyperexponential.

Hyperexponential Distribution (cont.)

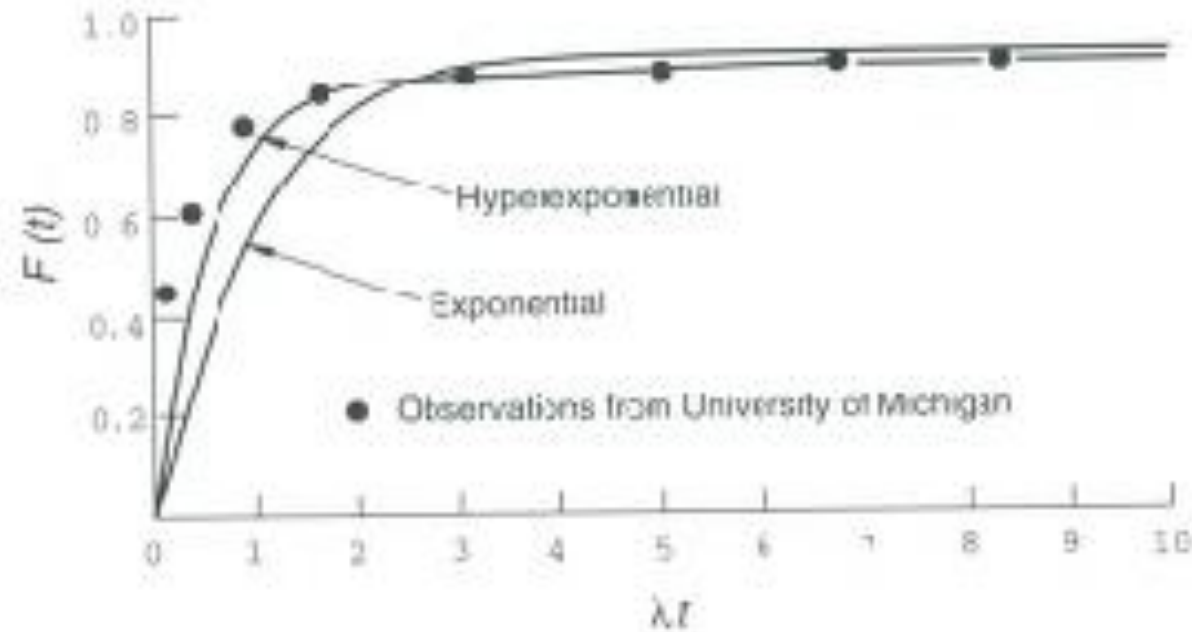

Figure 3.13. The CPU service time distribution compared with the hyperexponential distribution. (Reproduced from R. F. Rosin, "Determining a computing center environment," *CACM*, 1965; reprinted with permission of the Association of Computing Machinery.)

Example 3 (On-line Fault Detector)

- Consider a model consisting of a functional unit (e.g., an adder) together with an on-line fault detector
- Let T and C denote the times to failure of the unit and the detector.
- After the unit fails, a finite time D (called the *detection latency*) is required to detect the failure.
- Failure of the detector, however, is detected instantaneously.

Example 3 (On-line Fault Detector) cont.

- Let X denote the time to failure indication and Y denote the time to failure occurrence (of either the detector or the unit).
- Then

$$X = \min\{T + D, C\} \qquad \text{and} \qquad Y = \min\{T, C\}.$$

- If the detector fails before the unit, then a false alarm is said to have occurred.
- If the unit fails before the detector, then the unit keeps producing erroneous output during the detection phase and thus propagates the effect of the failure.
- The purpose of the detector is to reduce the detection time D.

Example 3 (On-line Fault Detector) cont.

- We define:

  Real reliability $R_r(t) = P(Y \geq t)$ and

  Apparent reliability $R_a(t) = P(X \geq t)$.

- A powerful detector will tend to narrow the gap between $R_r(t)$ and $R_a(t)$.
- Assume that *T, D,* and *C* are mutually independent and exponentially distributed with parameters λ, δ, and α.

Example 3 (On-line Fault Detector) cont.

- Then *Y* is *exponentially* distributed with parameter $\lambda + \alpha$ and:

$$R_r(t) = e^{-(\lambda+\alpha)t}$$

- *T* + *D* is *hypoexponentially* distributed so that:

$$F_{T+D}(t) = 1 - \frac{\delta}{\delta - \lambda} e^{-\lambda t} + \frac{\lambda}{\delta - \lambda} e^{-\delta t}$$

Example 3 (On-line Fault Detector) cont.

- And, the apparent reliability is:

$$
\begin{aligned}
R_a(t) &= P(X \geq t) \\
&= P(\min\{T + D, C\} \geq t) \\
&= P(T + D \geq t \text{ and } C \geq t) \\
&= P(T + D \geq t)P(C \geq t) \qquad \text{by independence} \\
&= [1 - F_{T+D}(t)]e^{-\alpha t} \\
&= \frac{\delta}{\delta - \lambda} e^{-(\lambda+\alpha)t} - \frac{\lambda}{\delta - \lambda} e^{-(\delta+\alpha)t}
\end{aligned}
$$

Phase-type Exponential Distributions

- Exponential Distribution:
  - Time to the event or Inter-arrivals => Poisson

- **Phase-type Exponential Distributions:**
- We have a process that is divided into *k* sequential phases, in which time that the process spends in each phase is:
  - Independent
  - Exponentially distributed

– The generalization of the phase-type exponential distributions is called **Coxian Distribution**
  - Any distribution can be expressed as the sum of phase-type exponential distributions

Summary

Four special types of phase-type exponential distributions:

1) Hypoexponential Distribution:

- Exponential distributions at each phase have different λ

2) K-stage Erlang Distribution:

- Exponential distributions in each phase are identical (with same λ)
- The number of phases (α) is an <u>integer</u>

3) Gamma Distribution

- Is a K-stage Erlang
- But the number of phases (α) is <u>not an integer</u>

4) Hyperexponential Distribution:

- A mixture of different exponential distributions