Hypoexponential, Erlang, and Gamma Distributions, TMR/Simplex

ECE 313

Probability with Engineering Applications

Lecture 21

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Announcements

- The examples shown on the board will not be necessarily posted online, so it is best if you take notes in the class.

- There will be weekly quizzes from the next week on the examples shown in the class, homeworks, and in-class projects.

- **Quiz 1: next Tuesday, November 12**
  - Topics covered in Lectures 18-19:
    - Covariance and Limit Theorems (Inequalities and CLT)
  - Homework 8 (Solutions will be posted today)

Today's Topics

- **Hypo**exponential Distribution
- TMR/Simplex Example
- Erlang and Gamma Distributions
- Standby Systems Example

Hypoexponential Distribution

- Many processes in nature can be divided into sequential phases.
- If the time the process spends in each phase is:
  - Independent
  - Exponentially distributed

  It can be shown that the overall time is ***hypoexponentially*** distributed.

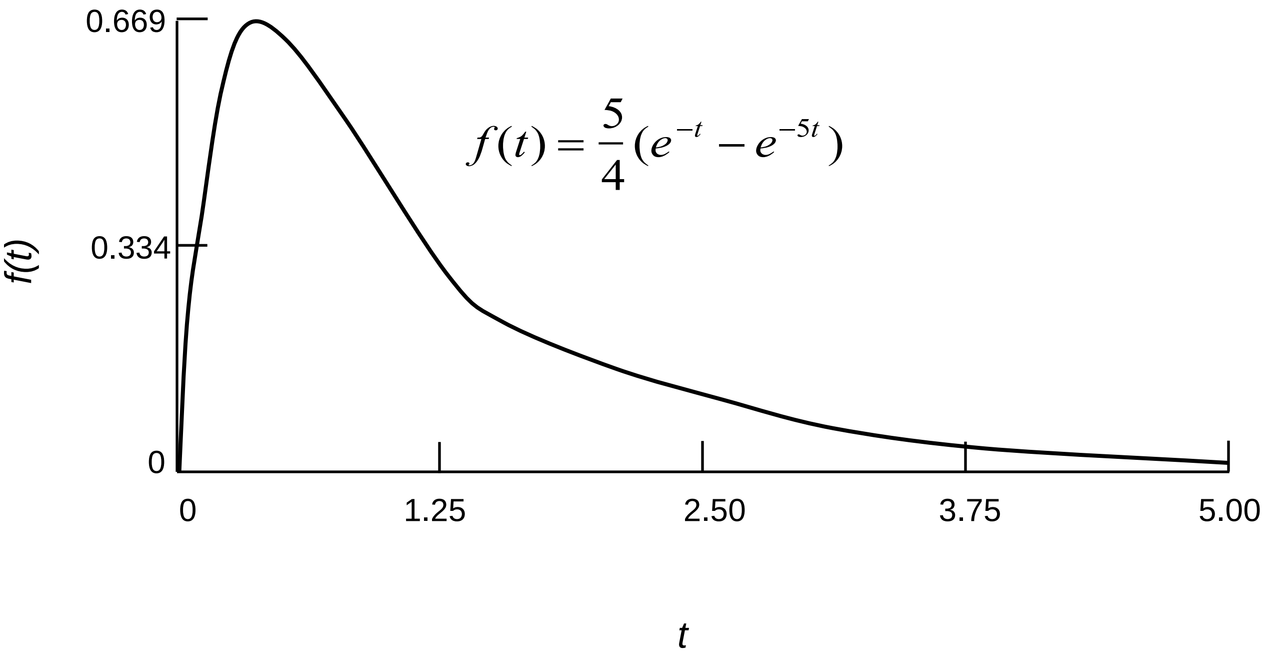
- For example: The service times for input-output operations in a computer system often possess this distribution.
- The distribution has r parameters, one for each distinct phase.

Hypoexponential Distribution (cont.)

- A two-stage hypoexponential random variable, *X*, with parameters λ_1 and λ_2 ($\lambda_1 \neq \lambda_2$), is denoted by X~HYPO(λ_1, λ_2).

$$f(t) = \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1}(e^{-\lambda_1 t} - e^{-\lambda_2 t}), \qquad t > 0$$

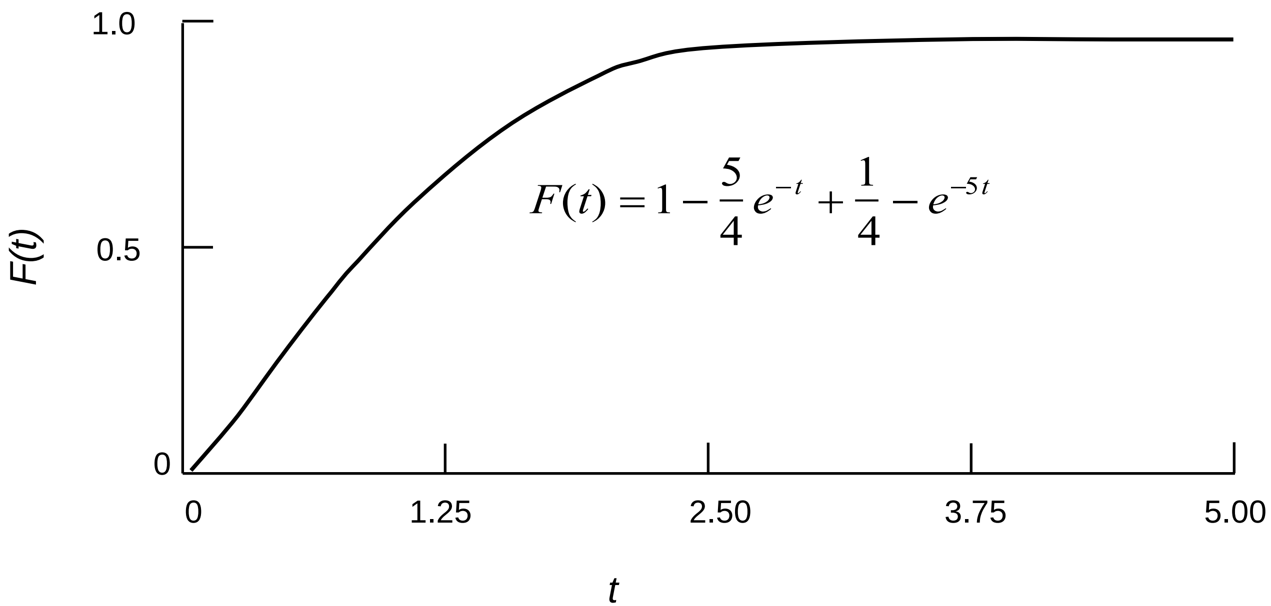
The pdf of the Hypoexponential Distribution



The CDF of the Hypoexponential Distribution

- The corresponding distribution function is:

$$F(t) = 1 - \frac{\lambda_2}{\lambda_2 - \lambda_1} e^{-\lambda_1 t} + \frac{\lambda_1}{\lambda_2 - \lambda_1} e^{-\lambda_2 t}, \qquad t \geq 0$$



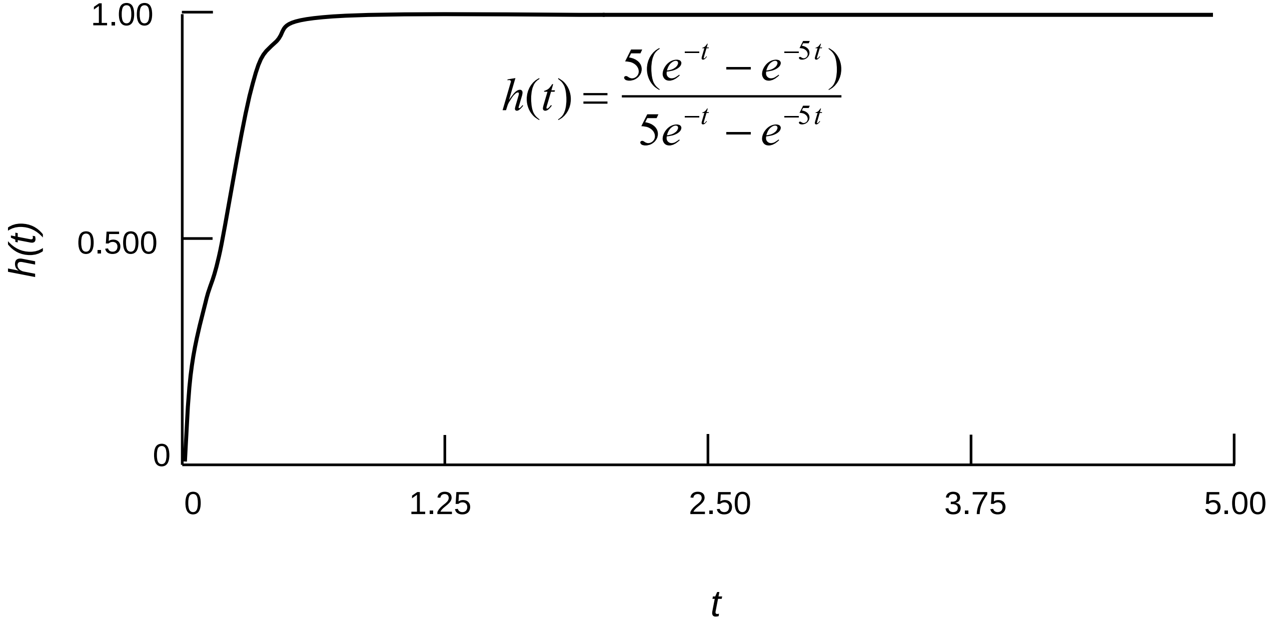
The Hazard for the Hypoexponential Distribution

- The hazard rate is given by:

$$h(t) = \frac{\lambda_1 \lambda_2 (e^{-\lambda_1 t} - e^{-\lambda_2 t})}{\lambda_2 e^{-\lambda_1 t} - \lambda_1 e^{-\lambda_2 t}}$$

The Failure Rate of the Hypoexponential Distribution

- This is an IFR distribution with the failure rate increasing from 0 up to min $\{\lambda_1, \lambda_2\}$



$$h(t) = \frac{5(e^{-t} - e^{-5t})}{5e^{-t} - e^{-5t}}$$

Example 1 (TMR system)

- Consider the triple modular redundant (TMR) system (n = 3 and m = 2).
- Let assume that R denotes the reliability of a component then
  - for the discrete time we assume that R = constant and then:

$$R_{TMR} = 3R^2 - 2R^3$$

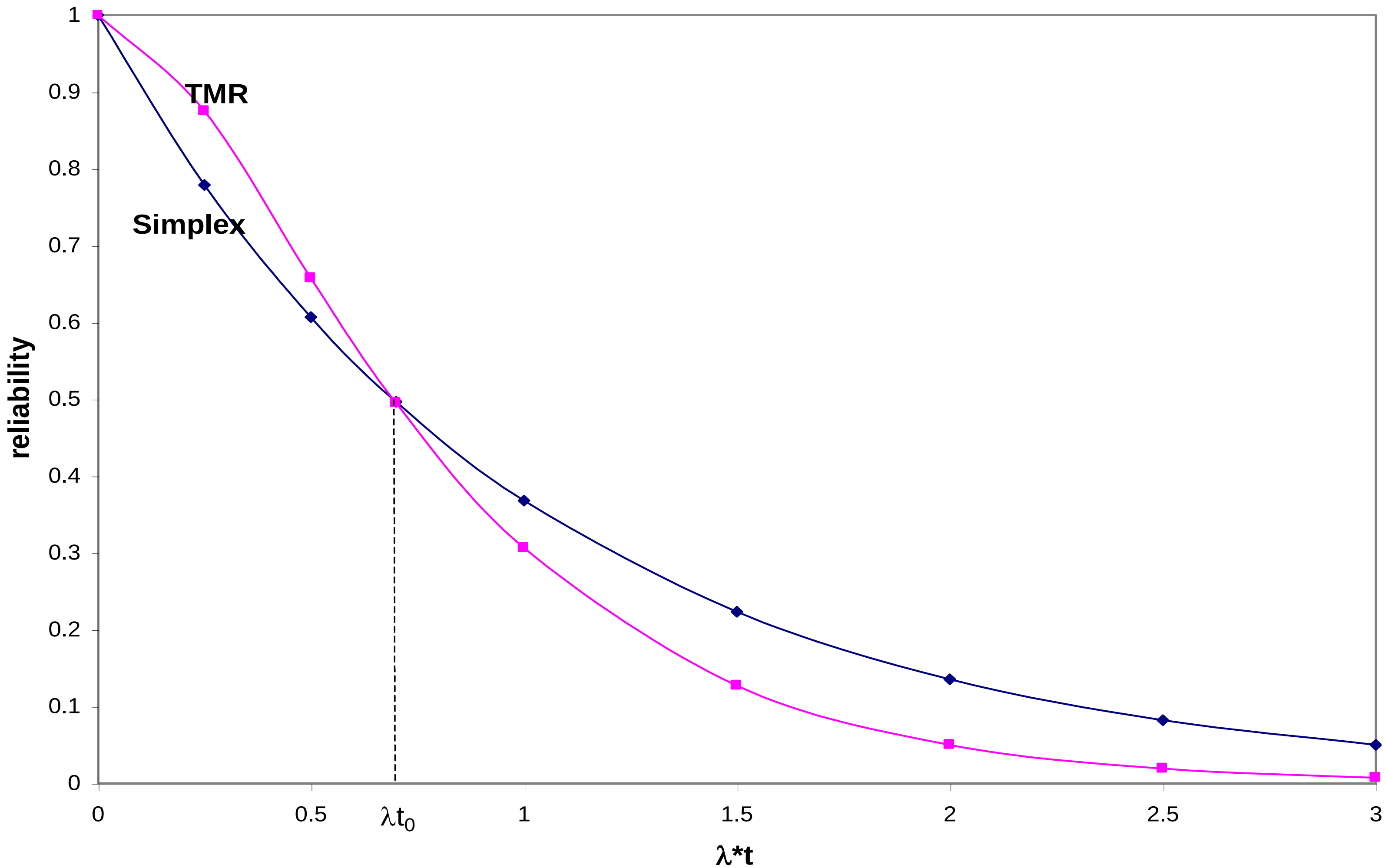
  - for the continuous time we assume that the failure of components are exponentially distributed with parameter λ
    - the reliability of a component can be expressed as:

$R(t) = 1 - (1 - e^{-\lambda t}) = e^{-\lambda t}$, and consequently

$$R_{TMR}(t) = 3R(t)^2 - 2R(t)^3$$

$$R_{TMR}(t) = 3\, e^{-2\lambda t} - 2\, e^{-3\lambda t}$$

Comparison of TMR and Simplex



Example 1 (TMR system) cont.

- From the plot $R_{tmr}(t)$ against t as well as $R(t)$ against t, note that:

$$R_{TMR}(t) \geq R(t), \qquad 0 \leq t \leq t_0,$$

and

$$R_{TMR}(t) \leq R(t), \qquad t_0 \leq t \leq \infty,$$

- Where t_0 is the solution to the equation:

$$3e^{-2\lambda t_0} - 2e^{-3\lambda t_0} = e^{-\lambda t_0},$$

which is:

$$t_0 = \frac{1n2}{\lambda} \approx \frac{0.7}{\lambda}.$$

Example 1 (TMR system) cont.

- Thus if we define a “short” mission by the mission time $t \leq t_0$, then it is clear that TMR type of redundancy improves reliability only for short missions.

- For long missions, this type of redundancy actually degrades reliability.

Example 2 (TMR/Simplex system)

- TMR system has higher reliability than simplex for short missions only.
- To improve upon the performance of TMR, we observe that after one of the three units have failed, both of the two remaining units must function properly for the classical TMR configuration to function properly.
- Thus, after one failure, the system reduces to a series system of two components, from the reliability point of view.

Example 2 (TMR/Simplex system) cont.

- An improvement over this simple scheme, known as TMR/simplex
  - detects a single component failure,
  - discards the failed component, and
  - reverts to one of the nonfailing simplex components.
- In other words, not only the failed component but also one of the good components is discarded.

Example 2 (TMR/Simplex system) cont.

- Let X, Y, Z, denote the times to failure of the three components.
- Let W denote the residual time to failure of the selected surviving component.
- Let X, Y, Z be mutually independent and exponentially distributed with parameter λ.
- Then, if L denotes the time to failure of TMR/simplex, then L can be expressed as:

$$L = \min\{X, Y, Z\} + W$$

Example 2 (TMR/Simplex system) cont.

- We use two useful notions:

1. The lifetime distribution of a series system whose components have independent exponentially distributed lifetimes is itself *exponentially distributed* with parameter $\sum_{i=1}^{n} \lambda_i$

2. If X´ and Y´are

- exponentially distributed with parameters λ_1 and λ_2, respectively,
- X´ and Y´ are independent , and
- $\lambda_1 \neq \lambda_2$ then

$$Z = X' + Y'$$

has a *two-stage hypoexponential* distribution with parameters λ_1 and λ_2,

Example 2 (TMR/Simplex system) cont.

- Now:
  - Since the exponential distribution is memoryless, the lifetime W of the surviving component is *exponentially distributed* with parameter λ.
  - min $\{X, Y, Z\}$ is *exponentially distributed* with parameter 3λ.
  - Then L has a *two-stage hypoexponential* distribution with parameters 3λ and λ.

Example 2 (TMR/Simplex system) cont.

- Therefore, we have:

$$F_L(t) = 1 - \frac{3\lambda}{2\lambda} e^{-\lambda t} + \frac{\lambda}{2\lambda} e^{-3\lambda t}, \qquad t \geq 0$$

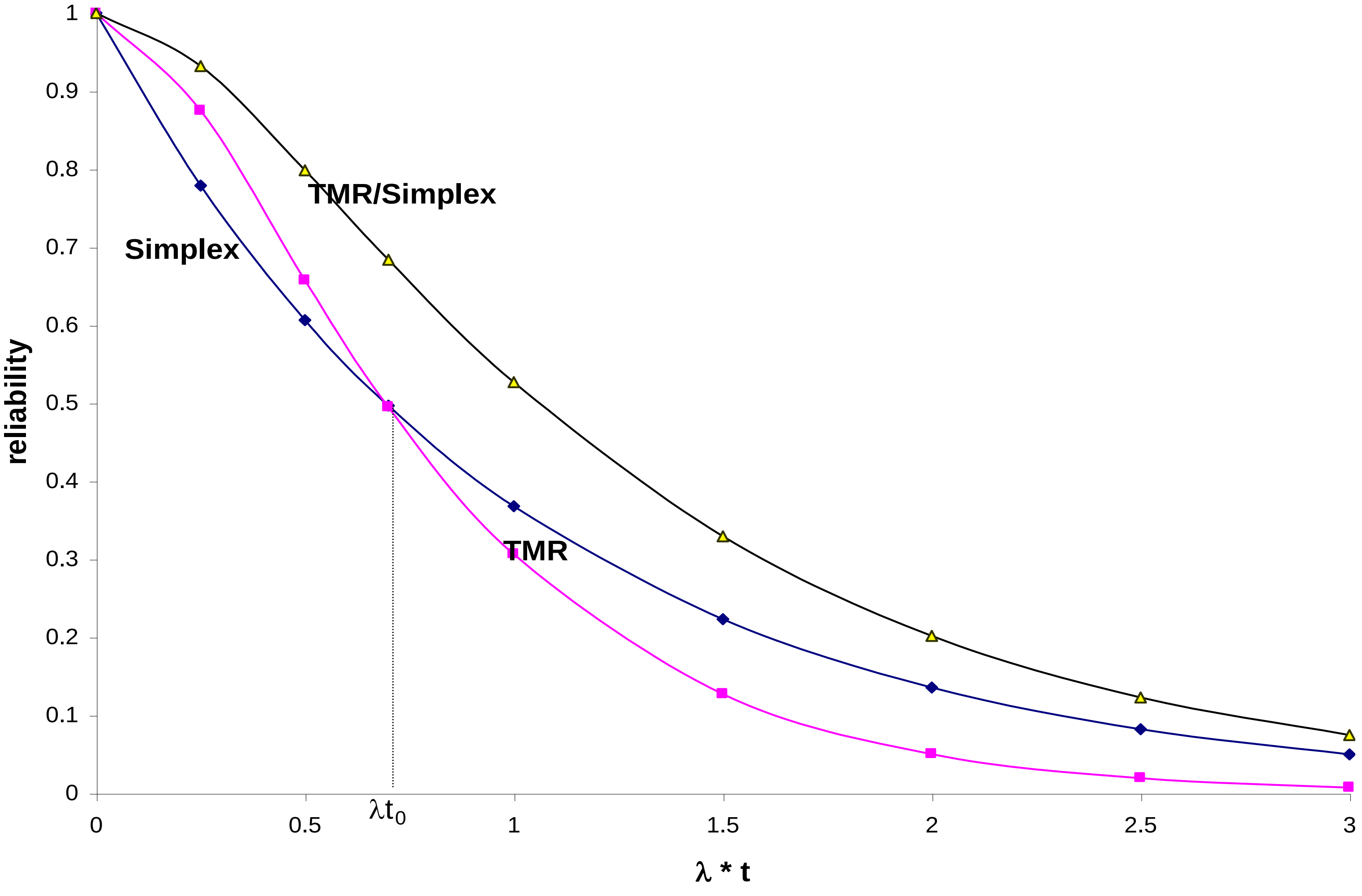
$$= 1 - \frac{3e^{-\lambda t}}{2} + \frac{e^{-3\lambda t}}{2}$$

- The reliability expression of TMR/simplex is:

$$\mathrm{R(t)} = \frac{3e^{-\lambda t}}{2} + \frac{e^{-3\lambda t}}{2}$$

- The TMR/simplex has a higher reliability than either a simplex or an ordinary TMR system for all $t \geq 0$.

Comparison of Simplex, TMR, & TMR/Simplex



Erlang and Gamma Distribution

- When r sequential phases have independent identical exponential distributions, the resulting density is known as r-stage (or r-phase) Erlang:

$$f(t) = \frac{\lambda^r t^{r-1} e^{-\lambda t}}{(r-1)!}, \qquad t > 0, \lambda > 0, r = 1, 2, \ldots$$

- The distribution function is:

$$F(t) = 1 - \sum_{k=0}^{r-1} \frac{(\lambda t)^k}{k!} \; e^{-\lambda t}, \qquad t \leq 0, \lambda > 0, r = 1, 2, \ldots$$

- Also:

$$h(t) = \frac{\lambda^r t^{r-1}}{(r-1)! \sum_{k=0}^{r-1} \frac{(\lambda t)^k}{k!}}, \qquad t > 0, \lambda > 0, r = 1, 2, \ldots$$

Erlang and Gamma Distribution (cont.)

- The exponential distribution is a special case of the Erlang distribution with $r = 1$.
- A component subjected to an environment so that N_t, the number of peak stresses in the interval $(0, t]$, is Poisson distributed with parameter λt.
- The component can withstand $(r - 1)$ peak stresses and the rth occurrence of a peak stress causes a failure.
- The component lifetime X is related to N_t so these two events are equivalent:

$$[X > t] = [N_t < r]$$

Erlang and Gamma Distribution (cont.)

- Thus:

$$\begin{aligned} R(t) &= P(X > t) \\ &= P(N_t < r) \\ &= \sum_{k=0}^{r-1} P(N_t = k) \\ &= e^{-\lambda t} \sum_{k=0}^{r-1} \frac{(\lambda t)^k}{k!} \end{aligned}$$

- *F(t)* = 1 - *R(t)* yields the previous formula:

$$F(t) = 1 - \sum_{k=0}^{r-1} \frac{(\lambda t)^k}{k!}\ e^{-\lambda t}, \qquad t \le 0, \lambda > 0, r = 1, 2, \ldots$$

- Conclusion is that the component lifetime has an *r*-stage Erlang distribution.

Gamma Function and Density

- If *r* (call it α) take nonintegral values, then we get the gamma density:

$$f(t) = \frac{\lambda^{\alpha} t^{\alpha-1} e^{-\lambda t}}{\Gamma\alpha}, \qquad \alpha > 0, t > 0$$

where the **gamma function** is defined by the integral:

$$\Gamma\alpha = \int_0^{\infty} x^{\alpha-1} e^{-x} dx, \qquad \alpha > 0$$

Gamma Function and Density (cont.)

- The following properties of the gamma function will be useful in the sequel. Integration by parts shows that for $\alpha > 1$:

$$\Gamma\alpha = (\alpha - 1)\Gamma(\alpha - 1)$$

- In particular, if α is a positive integer, denoted by *n*, then:

$$\Gamma n = (n-1)!$$

- Other useful formulas related to the gamma function are:

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

and

$$\int_0^\infty x^{\alpha-1} e^{-\lambda x} dx = \frac{\Gamma\alpha}{\lambda^\alpha}$$

Gamma Function and Density (cont.)

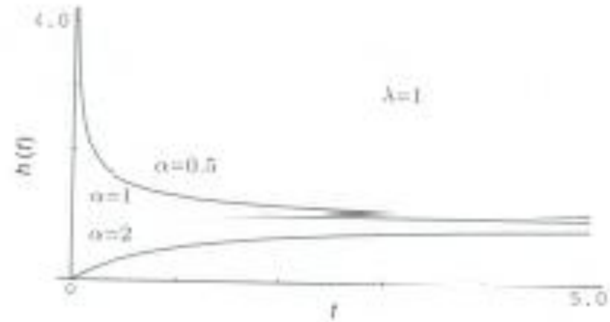


Figure 3.11. The failure rate of the gamma distribution

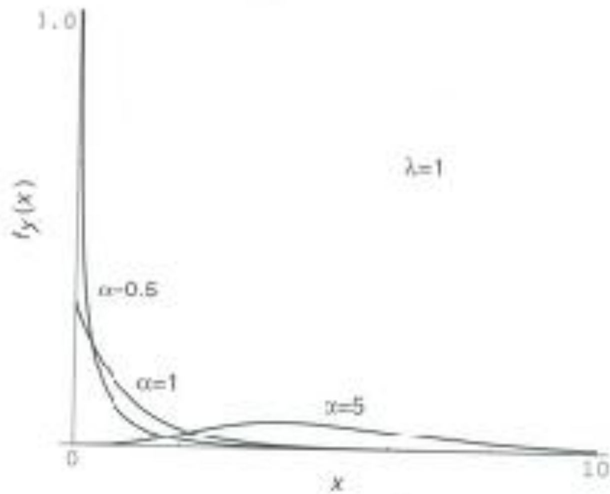


Figure 3.12. The gamma pdf