Hazard and Reliability Functions, Failure Rates

ECE 313

Probability with Engineering Applications

Lecture 20

Professor Ravi K. Iyer

Dept. of Electrical and Computer Engineering

University of Illinois at Urbana Champaign

Announcements

- The examples shown on the board will not be necessarily posted online, so it is best if you take notes in the class.

- There will be weekly quizzes from the next week on the examples shown in the class, homeworks, and in-class projects.

- **Quiz 1: next Tuesday, November 12**
  - Topics covered in Lectures 18-19:
    - Covariance and Limit Theorems (Inequalities and CLT)
  - Homework 8 (Solutions will be posted this Thursday)

Today's Topics

- Review of Joint and Conditional Density Functions
- Hazard Function
- Reliability Function
- Instantaneous Failure Rate
- Examples

Instantaneous Failure Rate or Hazard Rate

- Hazard measures the conditional probability of a failure given the system is currently working.
- The failure density (pdf) measures the overall speed of failures
- The Hazard/Instantaneous Failure Rate measures the dynamic (instantaneous) speed of failures.
- To understand the hazard function we need to review conditional probability and conditional density functions (very similar concepts)

Review of Joint and Conditional Density Functions

- We define the joint density function for two continuous random variables X and Y by:

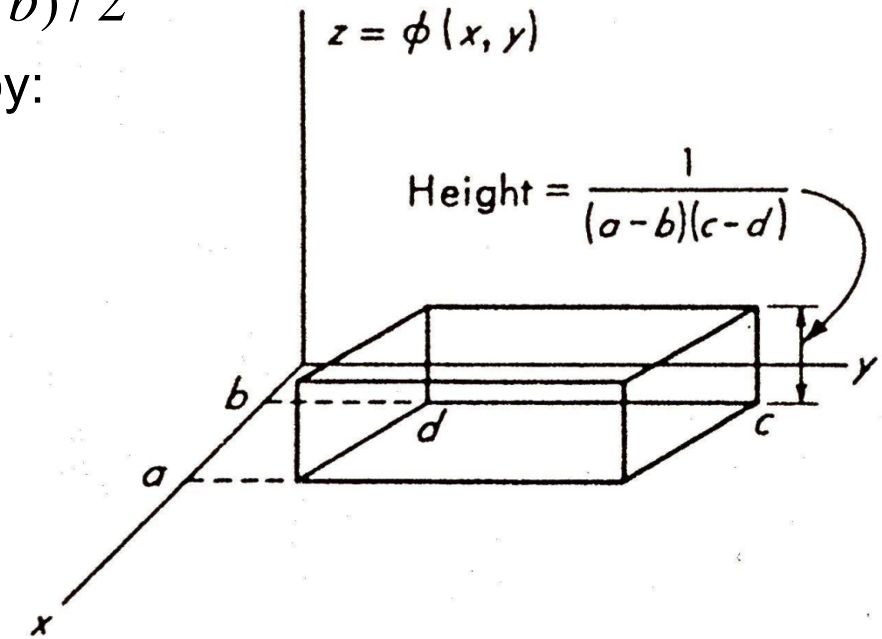
$$\phi(x, y)dxdy = P(x < X \le x + dx, y < Y \le y + dy)$$

- The cumulative distribution function associated with this density function is given by:

$$\Phi(x, y) = \int_{-\infty}^{x}\int_{-\infty}^{y} \phi(x, y)\, dx\, dy = P(-\infty < X \le x, -\infty < Y \le y)$$

Parallelepiped Density Function

- A parallelepiped density function is shown below:
- The probability that $b < X \le (a+b)/2$ and $d < Y \le (c+d)/2$ is given by:



$$P(b < X \le \frac{a+b}{2}, d < Y \le \frac{c+d}{2})$$

$$= \Phi(\frac{a+b}{2}, \frac{c+d}{2})$$

$$= \int_a^{(a+b)/2} \int_d^{(c+d)/2} \frac{1}{(a-b)(c-d)} dy\, dx$$

$$= \frac{1}{(a-b)(c-d)} (\frac{c+d}{2} - d)(\frac{a+b}{2} - b) = \frac{1}{4}$$

Review of Joint and Conditional Density Functions (Cont'd)

- $P(AB) = P(A)P(B)$
- Now if we associate random variable X with A and random variable Y with B then,

  $$P(x < \mathrm{x} \le x + dx) = f(x)dx$$

  $$P(y < \mathrm{y} \le y + dy) = g(y)dy$$

  $$P(x < \mathrm{x} \le x + dx, y < \mathrm{y} \le y + dy) = \phi(x, y)dx\,dy$$

  Thus joint probability $\phi(x, y) = f(x)g(y)$

- Remember random variables *X* and *Y* are independent if their joint density function is the product of the two marginal density functions.

Review of Joint and Conditional Density Functions (Cont'd)

- If events A and B are not independent, we must deal with *dependent or conditional* probabilities. Recall the following relations $P(AB) = P(A)P(B|A)$

$$P(B|A) = \frac{P(AB)}{P(A)}$$

- We can express conditional probability in terms of the random variables X and Y

$$P(y < \mathrm{Y} \le y + dy \,|\, x < \mathrm{X} \le x + dx) = \frac{P(x < \mathrm{X} \le x + dx, y < \mathrm{Y} \le y + dy)}{P(x < \mathrm{X} \le x + dx)}$$

- The left-hand side defines the conditional density function for y given x, which is written as $h(y|x) = \dfrac{\phi(x, y)}{f(x)}$

Review of Joint and Conditional Density Functions (Cont'd)

- Similarly, the conditional density function for x given y is

$$w(x|y) = \frac{\phi(x, y)}{g(y)}$$

- Now we use this to determine hazard function as a conditional density function

Hazard Function: From Conditional Density Definition

- The time to failure of a component is the random variable T. Therefore the failure density function is defined by

$$P(t < \mathrm{T} \le t + dt) = f(t)dt$$

- Sometimes it is more convenient to deal with the probability of failure between time t and $t+dt$, given that there were no failures up to time t. The probability expression becomes

$$P(t < \mathrm{T} \le t + dt \mid \mathrm{T} > t) = \frac{P(t < \mathrm{T} \le t + dt)}{P(\mathrm{T} > t)} \qquad \text{(a)}$$

where $P(\mathrm{T} > t) = 1 - P(\mathrm{T} < t) = 1 - F(t)$

Hazard Function

- The conditional probability on the left side (a) gives rise to the conditional probability function $z(t)$ defined by

$$z(t) = \lim_{dt \to 0} \frac{P(t < \mathrm{T} \le t + dt \mid T > t)}{dt} \quad \text{(b)}$$

- The conditional function is generally called the *hazard.* Combining (a) and (b):

$$z(t) = \frac{f(t)}{1 - F(t)}$$

- The main reason for defining the $z(t)$ function is that it is often more convenient to work with than $f(t)$.

Hazard Function

- For example, suppose that *f(t)* is an exponential distribution, the most common failure density one deals with in reliability work.

$$f(t) = \lambda e^{-\lambda t}$$

$$F(t) = 1 - e^{-\lambda t}$$

$$1 - F(t) = e^{-\lambda t}$$

$$z(t) = \frac{f(t)}{1 - F(t)} = \frac{\lambda e^{-\lambda t}}{e^{-\lambda t}} = \lambda$$

- Thus, an exponential failure density corresponds to a constant hazard function.
- What are the implications of this result?

The Reliability Function

- Let the random variable X be the lifetime or the time to failure of a component. The probability that the component survives until some time *t* is called the **reliability** *R(t)* of the component:

$$R(t) = P(X > t) = 1 - F(t)$$

where F is the distribution function of the component lifetime, X.

- The component is assumed to be working properly at time t=0 and no component can work forever without failure:

i.e. $$R(0) = 1 \quad and \quad \lim_{t \to \infty} R(t) = 0$$

- R(t) is a monotone non-increasing function of t.
- For t less than zero, reliability has no meaning, but: sometimes we let R(t)=1 for t<0. F(t) will often be called the **unreliability**.

The Reliability Function (Cont'd)

- Consider a fixed number of identical components, N_0, under test.
- After time t, $N_f(t)$ components have failed and $N_s(t)$ components have survived

$$N_f(t) + N_s(t) = N_0$$

- The estimated probability of survival:

$$\hat{P}(survival) = \frac{N_s(t)}{N_0}$$

The Reliability Function (Cont'd)

- In the limit as $N_0 \to \infty$, we expect $\hat{P}$ (survival) to approach $R(t)$. As the test progresses, $N_s(t)$ gets smaller and $R(t)$ decreases.

$$R(t) \approx \frac{N_s(t)}{N_0}$$

$$= \frac{N_0 - N_f(t)}{N_0}$$

$$= 1 - \frac{N_f(t)}{N_0}$$

The Reliability Function (Cont'd)

- (N_0 is constant, while the number of failed components N_f increases with time.)
- Taking derivatives: $$R'(t) \approx -\frac{1}{N_0} N'_f(t)$$

  $N'_f(t)$ is the rate at which components fail
- As $N_0 \rightarrow \infty$, the right hand side may be interpreted as the negative of the failure density function, $F_X(t)$

$$R'(t) = -f_x(t)$$

- Note: $f(t)\Delta t$ is the (unconditional) probability that a component will fail in the interval $(t, t + \Delta t)$

Instantaneous Failure Rate

- If we know for certain that the component was functioning up to time t, the (conditional) probability of its failure in the interval will (in general) be different from $f(t)\Delta t$

- This leads to the notion of "Instantaneous failure rate." Notice that the conditional probability that the component does not survive for an (additional) interval of duration x given that it has survived until time t can be written as:

$$G_Y(x \mid t) = \frac{P(t < X < t + x)}{P(X > t)} = \frac{F(t+x) - F(t)}{R(t)}$$

Instantaneous Failure Rate (Cont'd)

- Definition: The instantaneous failure rate $h(t)$ at time t is defined to be:

$$h(t) = \lim_{x \to 0} \frac{1}{x} \frac{F(t+x) - F(t)}{R(t)} = \lim_{x \to 0} \frac{R(t) - R(t+x)}{xR(t)}$$

so that:

$$h(t) = \frac{f(t)}{R(t)}$$

- $h(t)\Delta t$ represents the conditional probability that a component surviving to age t will fail in the interval $(t, t+\Delta t)$.
- The exponential distribution is characterized by a constant instantaneous failure rate:

$$h(t) = \frac{f(t)}{R(t)} = \frac{\lambda e^{-\lambda t}}{e^{-\lambda t}} = \lambda$$

Instantaneous Failure Rate (Cont'd)

- Integrating both sides of the equation:

$$\int_0^t h(x)dx = \int_0^t \frac{f(x)}{R(x)}dx$$

$$= \int_0^t -\frac{R'(x)}{R(x)}dx$$

$$= -\int_{R(0)}^{R(t)} \frac{dR}{R}$$

or $$-\ln R(t) = \int_0^t h(x)dx$$

(Using the boundary condition, R(0)=1) Hence:

$$R(t) = \exp\left[-\int_0^t h(x)dx\right]$$

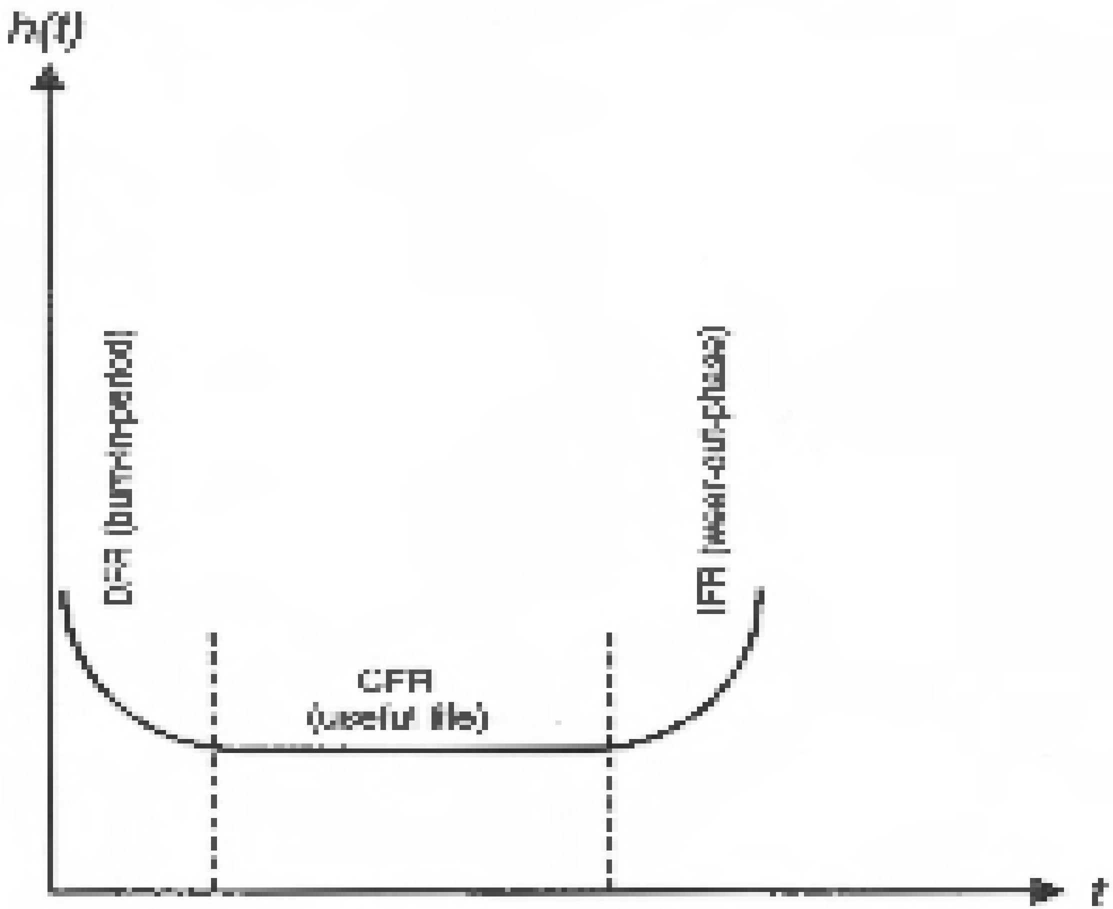
Cumulative Hazard

- The cumulative failure rate, $H(t) = \int_0^t h(x)dx$, is referred to as the **cumulative hazard.**
- $R(t) = \exp\left[-\int_0^t h(x)dx\right]$ gives a useful theoretical representation of reliability as a function of the failure rate.
- An alternate representation gives the reliability in terms of cumulative hazard: $R(t) = e^{-H(t)}$
- If the lifetime is exponentially distributed, then $H(t) = \lambda t$ and we obtain the exponential reliability function.

f(t) and *h(t)*

- $f(t)\Delta t$ is the unconditional probability that the component will fail in the interval $(t, t+\Delta t]$
- $h(t)\,\Delta t$ is the conditional probability that the component will fail in the same time interval, given that it has survived until time t.
- h(t) is always greater than or equal to f(t), because $R(t) \leq 1$.
- f(t) is a probability density. h(t) is not.
- [h(t)] is the failure rate
- [f(t)] is the failure density.
- To further see the difference, we need the notion of conditional probability density.

Failure Rate as a Function of Time



Conditional Probability Density Function $V_X(x|t)$

- Let $V_X(x|t)$ denote the conditional distribution of the lifetime X given that the component has survived past fixed time t. Then

$$V_X(x|t) = \frac{\int_t^x f(y)dy}{P(X > t)}$$

$$= \begin{cases} \dfrac{F(x) - F(t)}{1 - F(t)}, & x \geq t, \\ 0, & x < t. \end{cases}$$

- Note that $V_X(x|t) = G_Y(x - t|t)$. Then the conditional failure density is:

$$v_X(x|t) = \begin{cases} \dfrac{f(x)}{1 - F(t)}, & x \geq t, \\ O, & x < t. \end{cases}$$

Conditional Probability Density Function $V_X(x|t)$ (Cont'd)

- The conditional density $v_X(x|t)$ satisfies properties (f1) and (f2) of a probability density function (pdf) and hence is a probability density while the failure rate $h(t)$ does not satisfy property (f2) since:

$$0 = \lim_{t \to \infty} R(t) = \exp\left[-\int_0^\infty h(t)dt \right] \neq 1$$

Treatment of Failure Data

- Part failure data generally obtained from two sources: the failure times of various items in a population placed on a life test, or repair reports listing operating hours of replaced parts in equipment already in field use.
- Compute and plot either the failure density function or the instantaneous failure rate as a function of time.
- The data: a sequence of times to failure, but the failure density function and the hazard introduced as continuous variables.
- Compute a piecewise-continuous failure density function and hazard rate from the data.
- This is, a specific approach to the very general engineering problem of how to model a problem from certain qualitative knowledge about the system supported by quantitative data.

Treatment of Failure Data (cont'd)

- Define piecewise-continuous failure density and hazard-rate in terms of the data.
- Assume that our data describe a set of N items placed in operation at time t=0. As time progresses, items fail, and at any time t the number of survivors in n(t).
- The empirical probability density function defined over the time interval $t_i < t \leq t_i + \Delta t_i$, is given by the ratio of the number of failures occurring in the interval to the *size of the original population*, divided by the length of the time interval:

$$f_d(t) = \frac{[n(t_i) - n(t_i + \Delta t_i)]/N}{\Delta t_i} \quad for\, t_i < t \leq t_i + \Delta t_i$$

Treatment of Failure Data (Cont'd)

- The data hazard (inst. failure rate) over the interval $t_i < t \le t_i + \Delta t_i$ is the ratio of the number of failures occurring in the time interval to the *number of survivors at the beginning of the time interval*, divided by the length of the time interval:

$$z_d(t) = \frac{[n(t_i) - n(t_i + \Delta t_i)]/n(t_i)}{\Delta t_i} \; for \; t_i < t \le t_i + \Delta t_i$$

- Observation: the failure density function $f_d(t)$ is a measure of the *overall speed* at which failures are occurring, whereas the hazard rate $z_d(t)$ is a measure of the *instantaneous speed* of failure.
- Note: both $f_d(t)$ and $z_d(t)$ have the dimensions of inverse time (generally the time unit is hours).
- The choice of t_i and Δt_i in the above equations is unspecified and is best discussed in terms of the examples that follow.

Properties of Density and Distribution Functions

No.	Distribution Function	Density Function
1.	$F(x)$ for $x_1 < x \leq x_2$ Distribution function defined over range $x_1 < x \leq x_2$	$f(x)$ for $x_1 < x \leq x_2$ Distribution function defined over range $x_1 < x \leq x_2$
2.	$P(a < x \leq b) = F(b) - F(a)$ Probability that x lies between a and b	$P(a < x \leq b) = \int_a^b f(x)dx$ Probability that x lies between a and b
3.	F(x) cannot decrease as x increases	$f(x) \geq 0$ f(x) is never negative
4.	$F(x_1) = 0$ and $F(x_2) = 1$. Probability ranges from 0 to 1	$\int_{x_1}^{x_2} f(x(dx) = 1$ Probability of the sample space is unity

Constraints on f(t) and z(t)

No.	Distribution Function	Density Function
1.	F(x) for $x_1 < x \leq x_2$ Distribution function defined over range $x_1 < x \leq x_2$	f(x) for $x_1 < x \leq x_2$ Distribution function defined over range $x_1 < x \leq x_2$
2.	$P(a < x \leq b) = F(b) - F(a)$ Probability that x lies between a and b	$P(a < x \leq b) = \int_a^b f(x)dx$ Probability that x lies between a and b
3.	F(x) cannot decrease as x increases	$f(x) \geq 0$ f(x) is never negative
4.	$F(x_1) = 0$ and $F(x_2) = 1$ Probability ranges from 0 to 1	$\int_{x_1}^{x_2} f(x(dx) = 1$ Probability of the sample space is unity

Hazard Rate Example

Table 4.1 Failure data for ten hypothetical electronic components

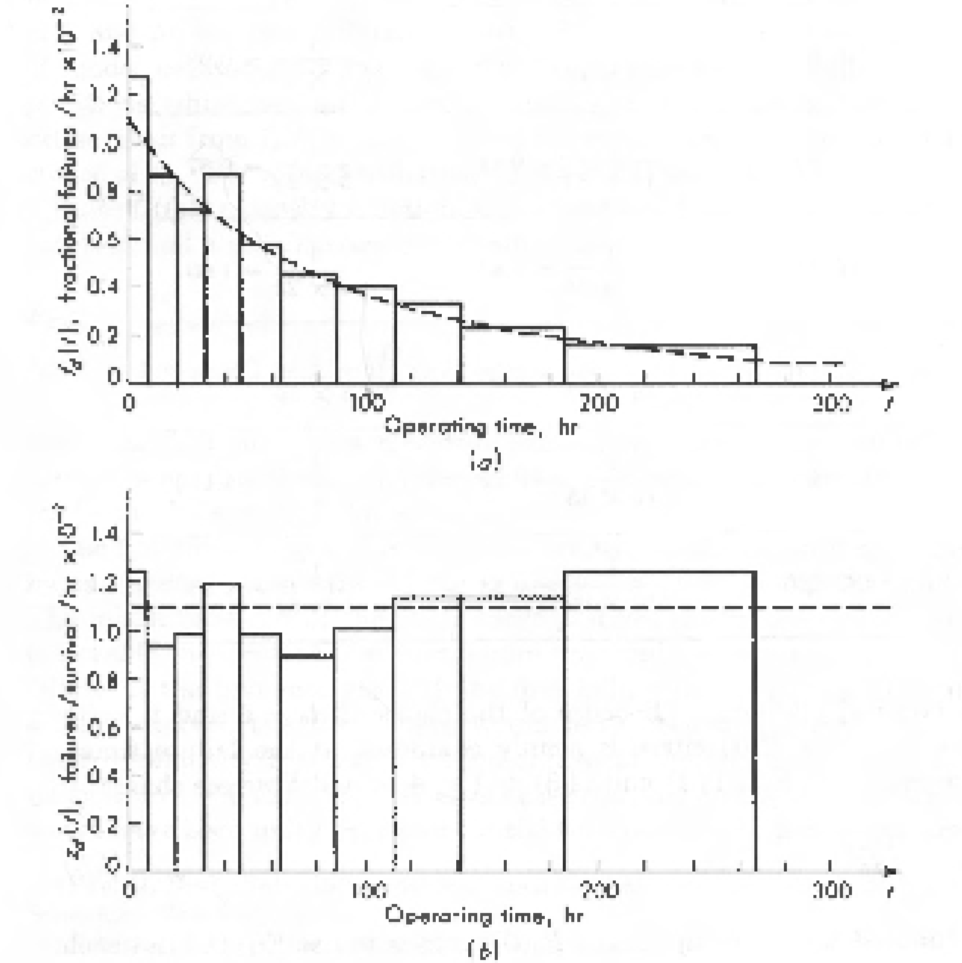
Failure number	*Operating time, hr*
1	8
2	20
3	34
4	46
5	63
6	86
7	111
8	141
9	186
10	266

Hazard Rate Example (Cont'd)

Table 4.2 Computation of data failure density and data hazard rate

Time interval, hr	*Failure density per hr* $f_d(t)(\times 10^{-2})$	*Hazard rate per hr* $z_d(t)(\times 10^{-2})$
0–8	$\frac{1}{10 \times 8} = 1.25$	$\frac{1}{10 \times 8} = 1.25$
8–20	$\frac{1}{10 \times 12} = 0.84$	$\frac{1}{9 \times 12} = 0.93$
20–34	$\frac{1}{10 \times 14} = 0.72$	$\frac{1}{8 \times 14} = 0.96$
34–46	$\frac{1}{10 \times 12} = 0.84$	$\frac{1}{7 \times 12} = 1.19$
46–63	$\frac{1}{10 \times 17} = 0.59$	$\frac{1}{6 \times 17} = 0.98$
63–86	$\frac{1}{10 \times 23} = 0.44$	$\frac{1}{5 \times 23} = 0.87$
86–111	$\frac{1}{10 \times 25} = 0.40$	$\frac{1}{4 \times 25} = 1.00$
111–141	$\frac{1}{10 \times 30} = 0.33$	$\frac{1}{3 \times 30} = 1.11$
141–186	$\frac{1}{10 \times 45} = 0.22$	$\frac{1}{2 \times 45} = 1.11$
186–266	$\frac{1}{10 \times 80} = 0.13$	$\frac{1}{1 \times 80} = 1.25$

Hazard Rate Example (Cont'd)



Failure Density and Hazard Rate for the Given Data