Exam Review, Limit Theorems, and Mini Project 3

ECE 313

Probability with Engineering Applications

Lecture 19

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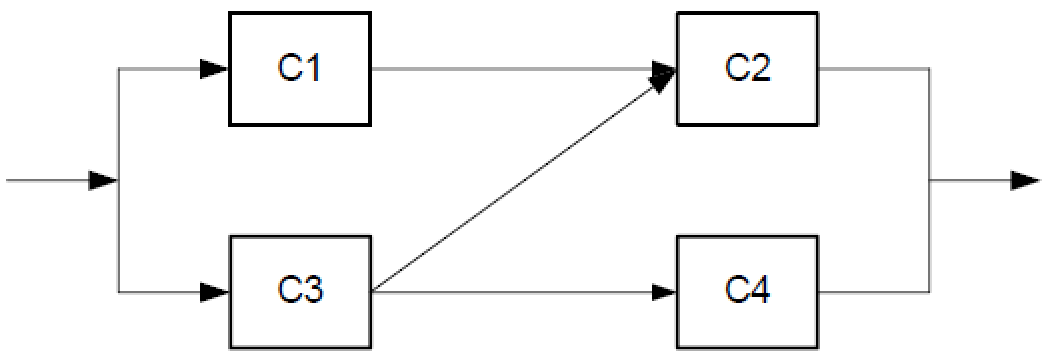
Today's Topics

- Review of Midterm Exam
- Limit Theorems
- Mini Project 3
  - Description, Tasks, and Due Dates

Midterm Exam Review: Problem 3.A

- Consider the non-series-parallel system of four independent components shown in the following figure. The system is considered to be functioning properly if all components along at least one path from input to output are functioning properly.



- **Part A:** Determine an expression for the system reliability as a function of component reliabilities (Assume that the reliability of each component is equal to *R).*

Midterm Exam Review: Problem 3 (Cont'd)

- **Part B:** Calculate the mean time to the failure (MTTF) of the system. Assume that the time to failure (lifetime) of each component is exponentially distributed with parameter λ.

- **Part C:** Recall the reliability expression for a TMR system (with a perfect voter) from your in-class project. Compare MTTF of the system shown here with MTTF of a TMR system composed of the same components (with reliability of *R and an exponential time to failure with parameter λ). Which one will fail on average earlier?*

Midterm Exam Review: Problem 3 (Solution)

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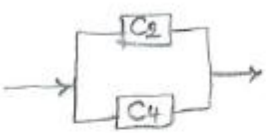
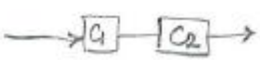
Part A (5 pts): Determine an expression for the system reliability as a function of component reliabilities (Assume that the reliability of each component is equal to R).

We condition on component C_3:

$$P(X) = P(X \mid X_3)\underbrace{P(X_3)}_{C_3 \text{ Working}} + P(X \mid \overline{X}_3)\underbrace{P(\overline{X}_3)}_{C_3 \text{ Not working}}$$



$$P(X \mid X_3) = 1 - (1-R_2)(1-R_4)$$



$$P(X \mid \overline{X}_3) = R_1 R_2$$

$$P(X_3) = R_3$$

$$P(\overline{X}_3) = (1-R_3)$$

$$\Rightarrow R_{sys} = R_3 \cdot [1-(1-R_2)(1-R_4)] + (1-R_3) \cdot R_1 R_2$$

$$= R\,[1-(1-R)^2] + (1-R)\,R^2$$

$$\boxed{R_{sys} = 3R^2 - 2R^3}$$

Midterm Exam Review: Problem 4.E

- Let the probability density of *X* be given by:

$$f(x) = \begin{cases} \frac{3}{8}(4x - 2x^2), & 0 < x < 2 \\ 0, & otherwise \end{cases}$$

- **Part E:** Let $Y = \sqrt{X+1}$. Find the probability distribution function (PDF) of Y.

Midterm Exam Review: Problem 4.E Solution

Part E (6 pts): Let $Y = \sqrt{X+1}$. Find the probability distribution function (PDF) of Y.

$$Y = \sqrt{X+1} \qquad 0 < x < 2 \quad \Rightarrow \quad 1 < y < \sqrt{3}$$

$$F_Y(y) = P(Y \leq y) = P(\sqrt{X+1} \leq y) = P(X+1 \leq y^2)$$

$$= P(X \leq y^2 - 1) = F_X(y^2 - 1)$$

From A $\longrightarrow$ $F_X(x) = \frac{3}{4}x^2 - \frac{1}{4}x^3$

$$\Rightarrow F_Y(y) = F_X(y^2-1) = \frac{3}{4}(y^2-1)^2 - \frac{1}{4}(y^2-1)^3$$

PDF of $Y \Rightarrow$ $f_Y(y) = \frac{dF_Y}{dy} = \frac{3}{4} 4y(y^2-1) - \frac{1}{4} 6y (y^2-1)^2, \quad 1 < y < \sqrt{3}$

Limit Theorems Example

- The importance of Markov's and Chebyshev's inequalities is that they enable us to derive bounds on probabilities when only the mean, or both the mean and the variance, of the probability distribution are known. If the actual distribution were known, then the desired probabilities could be exactly computed, and we would not need to resort to bounds.

- Example: Suppose we know that the number of items produced in a factory during a week is a random variable with mean 500.
  a) What can be said about the probability that this week's production will be at least 1000?
  b) If the variance of a week's production is known to equal 100, then what can be said about the probability that this week's production will be between 400 and 600?

Limit Theorems Example (Cont'd)

- Let X be the number of items that will be produced in a week.

  a) By Markov's inequality,

$$P\{X \geq 1000\} \leq \frac{E[X]}{1000} = \frac{500}{1000} = \frac{1}{2}$$

  b) By Chebyshev's inequality,

$$P\{|X - 500| \geq 100\} \leq \frac{\sigma^2}{(100)^2} = \frac{1}{100}$$

  Hence,

$$P\{|X - 500| < 100\} \leq 1 - \frac{1}{100} = \frac{99}{100}$$

  And so the probability that this week's production will be between 400 and 600 is at least 0.99.

Limit Theorems

- The ***strong law of large numbers*****,** is probably the most well-known result in probability theory. It states that the average of a sequence of independent random variables having the same distribution will, with probability 1, converge to the mean of that distribution.

- Let $X_1, X_2, \ldots$ be a sequence of independent random variables having a common distribution, and let $E[X_i] = \mu$ Then, with probability 1,

$$\frac{X_1 + X_2 + \ldots + X_n}{n} \to \mu \quad \text{as } n \to \infty$$

- As an example suppose that a sequence of independent trials is performed. Let E be a fixed event and denote P(E) the probability that E occurs on any particular trial. Letting

$$X_i = \begin{cases} 1, & \text{if } E \text{ occurs on the } i\text{th trial} \\ 0, & \text{if } E \text{ does not occur on the } i\text{th trial} \end{cases}$$

Limit Theorems (Cont'd)

- We have by the strong law of large numbers that, with probability 1,

$$\frac{X_1 + ... + X_n}{n} \to E\{X\} = P(E)$$

- Since $X_1 + ... + X_n$ represents the number of times that the event E occurs in the first n trials, we may interpret this equation as stating that, with probability 1, the limiting proportion of time that the event E occurs is just $P(E)$.
- Running neck and neck with the strong law of large numbers for the honor of being probability theory's number one result is the **central limit theorem.**
- The central limit theorem provides a simple method for computing approximate probabilities for sums of independent random variables.
- It also explains the remarkable fact that the empirical frequencies of so many natural "populations" exhibit a bell-shaped (that is, normal) curve.

Limit Theorems (Cont'd)

- **Central Limit Theorem:**
  Let $X_1, X_2, \ldots$ be a sequence of independent, identically distributed random variables, each with mean μ and variance σ^2 then the distribution of $\dfrac{X_1 + X_2 + \ldots + X_n - n\mu}{\sigma\sqrt{n}}$
- Tends to the standard normal as $n \to \infty$. That is,

$$P\left\{\frac{X_1 + X_2 + \ldots + X_n - n\mu}{\sigma\sqrt{n}} \le a\right\} \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-x^2/2} dx$$

as $n \to \infty$.

- Note that like other results, this theorem holds for any distribution of the X_i 's; herein lies its power.