Joint Distribution Functions, Independent Random Variables

ECE 313

Probability with Engineering Applications

Lecture 16

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Announcements

- **Midterm next Tuesday, October 22**

  11:00am – 12:20pm, in class
  - **All topics covered in Lectures 1 to 15**
  - **Homework 1-6, In-class projects 1-3, and Mini-Projects 1-2**

- **Be on time, exam starts at 11:00am sharp.**
- **You are allowed to bring only one 8”x11” sheet of notes**

- **Review Session Today, 5:00pm – 7:00pm, CSL 141**

- **Additional TA Office hours** on **Friday**, **2:00pm – 5pm, CSL 249.**

Today's Topics

- Quick Review on Joint Distribution Functions
  - Example
- Independence of Random Variables
- Review of Material for the Midterm Exam

Joint Distribution Functions

- We have concerned ourselves with the probability distribution of a single random variable
- Often interested in probability statements concerning two or more random variables
- Define, for any two random variables X and Y, the *joint cumulative probability distribution function* of X and Y by

$$F(a,b) = P\{X \le a, Y \le b\}, \ -\infty < a,b < \infty$$

- The distribution of X can be obtained from the joint distribution of X and Y as follows:

$$\begin{aligned} \mathrm{F_X}(a) &= P\{X \le a\} \\ &= P\{X \le a, Y < \infty\} \\ &= F(a,\infty) \end{aligned}$$

Joint Distribution Functions Cont'd

- Similarly, $F_Y(b) = P\{Y \leq b\} = F(\infty, b)$ Where X and Y are both discrete random variables it is convenient to define the *joint probability mass function* of X and Y by $p(x, y) = P\{X = x, Y = y\}$

- Probability mass function of X $p_X(x) = \sum_{y:p(x,y)>0} p(x, y)$

$$p_Y(y) = \sum_{x:p(x,y)>0} p(x, y)$$

- We say that X and Y are *jointly continuous* defined for all real x and y

$$P\{X \in A, Y \in B\} = \int_B \int_A f(x, y)dxdy$$

Joint Distribution Functions Cont'd

- Called the *joint probability density function* of X and Y. The probability density of X

$$P\{X \in A\} = P\{X \in A, Y \in (-\infty, \infty)\}$$

$$= \int_{-\infty}^{\infty} \int_A f(x, y)dxdy$$

$$= \int_A f_X(x)dx$$

$f_X(x) = \int_{-\infty}^{\infty} f(x, y)dy$ is thus the probability density function of X

- Similarly the probability density function of Y is $f_Y(y) = \int_{-\infty}^{\infty} f(x, y)dx$ because

$$F(a,b) = P(X \le a, Y \le b) = \int_{-\infty}^{a} \int_{-\infty}^{b} f(x, y)dydx$$

Joint Distribution Functions Cont'd

- Proposition: if X and Y are random variables and g is a function of two variables, then

$$E[g(X,Y)] = \sum_{y}\sum_{x} g(x,y)\,p(x,y)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\,f(x,y)\,dx\,dy$$

- For example, if $g(X,Y)=X+Y$, then, in the continuous case

$$E[X,Y] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x+y)\,f(x,y)\,dx\,dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xf(x,y)\,dx\,dy + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} yf(x,y)\,dx\,dy$$

$$= E[X] + E[Y]$$

Joint Distribution Functions Cont'd

- Where the first integral is evaluated by using the foregoing Proposition with $g(x,y)=x$ and the second with $g(x,y)=y$
- In the discrete case $E[aX+bY]=aE[X]+bE[Y]$
- Joint probability distributions may also be defined for n random variables. If $X_1, X_2, ..., X_n$ are n random variables, then for any n constants $a_1, a_2, ..., a_n$

$$E[a_1X_1+a_2X_2+...a_nX_n]=a_1E[X_1]+a_2E[X_2]+...+a_nE[X_n]$$

Example 3

Suppose that the joint probability mass function of X and Y is

$$P(X = i, Y = j) = \binom{j}{i} e^{-2\lambda} \lambda^j / j!, \quad 0 \le i \le j$$

(a) Find the probability mass function of Y.
(b) Find the probability mass function of X.
(c) Find the probability mass function of $Y - X$.

Example 3 (Cont'd)

a) Marginal PDF of *Y*:

$$\begin{aligned} P(Y=j) &= \sum_{i=0}^{j} \binom{j}{i} e^{-2\lambda} \lambda^j / j! \\ &= e^{-2\lambda} \frac{\lambda^j}{j!} \sum_{i=0}^{j} \binom{j}{i} 1^i 1^{j-i} \\ &= e^{-2\lambda} \frac{(2\lambda)^j}{j!} \end{aligned}$$

b) Marginal PDF of *X*:

$$\begin{aligned} P(X=i) &= \sum_{j=i}^{\infty} \binom{j}{i} e^{-2\lambda} \lambda^j / j! \\ &= \frac{1}{i!} e^{-2\lambda} \sum_{j=i}^{\infty} \frac{\lambda^j}{(j-i)!} \\ &= \frac{\lambda^i}{i!} e^{-2\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \\ &= e^{-\lambda} \frac{\lambda^i}{i!}. \end{aligned}$$

Example 3 (Cont'd)

c) We first calculate the joint density function of *X* and *Y*-*X*

$$\begin{aligned} P(X=i, Y-X=k) &= P(X=i, Y=k+i) \\ &= \binom{k+i}{i} e^{-2\lambda} \frac{\lambda^{k+i}}{(k+i)!} \\ &= e^{-2\lambda} \frac{\lambda^k}{k!} \frac{\lambda^i}{i!}. \end{aligned}$$

- Then summing up with respect to *i*, we get the marginal distribution of *Y – X*, which is for *k:*

$$\begin{aligned} P(Y-X=k) &= \sum_{i=0}^{\infty} P(X=i, Y-X=k) \\ &= e^{-2\lambda} \frac{\lambda^k}{k!} \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} \\ &= e^{-2\lambda} \frac{\lambda^k}{k!} e^{-\lambda} \\ &= e^{-\lambda} \frac{\lambda^k}{k!}. \end{aligned}$$

Independent Random Variables

- We define two random variables X and Y to be independent if:
$$F(x, y) = F_X(x)F_Y(y), \; -\infty < x < \infty, -\infty < y < \infty$$
- Independence of random variables X and Y implies that their joint CDF factors into the product of the marginal CDFs.
- Applies to all types of random variables
- In case X and Y are discrete, the preceding definition of independence is equivalent to $p(x, y) = P_X(x)P_Y(y)$
- If X and Y are continuous, the preceding definition of independence is equivalent to the condition
$$f(x, y) = f_X(x) f_Y(y), \; -\infty < x < \infty, -\infty < y < \infty$$
assuming that $f(x,y)$ exists.
- The joint distribution of X and Y when one of them is a discrete random variable while the other is a continuous random variable

Independent Random Variables Cont'd

- If X is discrete and Y is continuous their independence becomes:

$$P(X = x, Y \le y) = p_X(x) f_Y(y), \text{ all } x \text{ and } y$$

- The definition of joint distribution, joint density, and independence of two random variables can be easily generalized to a set of n random variables, $X_1, X_2, \ldots, X_n$.
- Example (Independent R.V.)
- Assume that the lifetime X and the brightness Y of a light bulb are being modeled as continuous random variables. Let the joint pdf be given by $f(x, y) = \lambda_1 \lambda_2 e^{-(\lambda_1 x + \lambda_2 y)}, 0 < x < \infty, 0 < y < \infty$
- This is known as the *bivariate exponential density*.
- The marginal density of X is $f_X(x) = \int_{-\infty}^{\infty} f(x, y)\, dy$

$$= \int_0^{\infty} \lambda_1 \lambda_2 e^{-(\lambda_1 x + \lambda_2 y)} dy$$

$$= \lambda_1 e^{-\lambda_1 x}, 0 < x < \infty$$

Independent Random Variables Cont'd

- Similarly $f_Y(y) = \lambda_2 e^{-\lambda_2 y}, 0 < y < \infty$
- It follows that X and Y are independent random variables:

$$f(xy) = f(x)f(y)$$

- The joint distribution function can be computed to be

$$F(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(u,v)\,dv\,du$$

$$= \int_{0}^{x} \int_{0}^{y} \lambda_1 \lambda_2 e^{-(\lambda_1 u + \lambda_2 v)} dv\,du$$

$$= (1 - e^{-\lambda_1 x})(1 - e^{-\lambda_2 y}), 0 < x < \infty, 0 < y < \infty.$$

Exam Review

- Basic Concepts:
  - ***Random experiment*** is an experiment the outcome of which is not certain
  - ***Sample Space (S)*** is the totality of the possible outcomes of a random experiment
  - ***Discrete (countable) sample space*** is a sample space which is either
    - *finite*, i.e., the set of all possible outcomes of the experiment is finite
    - *countably infinite, i.e.,* the set of all outcomes can be put into a one-to-one correspondence with the natural numbers
  - ***Continuous sample space*** is a sample space for which all elements constitute a continuum, such as all the points on a line, all the points in a plane
  - An ***event*** is a collection of certain sample points, i.e., a subset of the sample space
    - ***Universal event*** is the entire sample space S
    - *The null set* $\varnothing$ is a **null or impossible event**

Exam Review

- **Algebra of Events**
  - The ***intersection*** of E_1 and E_2 is given by:
    - $E_1 \cap E_2 = \{s \in S \mid s \text{ is an element of both } E_1 \text{ and } E_2\}$
  - The ***union*** E_1 and E_2 is given by:
    - $E_1 \cup E_2 = = \{s \in S \mid \text{either } s \in E_1 \text{ or } s \in E_2 \text{ or both}\}$
  - In general: $|E_1 \cup E_2| \leq |E_1| + |E_2|$
    - where $|A|$ = the number of elements in the set **(Cardinality)**
  - Definition of *union* and *intersection* extend to any finite number of sets:

$$\bigcup_{i=1}^{n} E_i = E_1 \cup E_2 \cup E_3 \cup \ldots \cup E_n$$

$$\bigcap_{i=1}^{n} E_i = E_1 \cap E_2 \cap E_3 \cap \ldots \cap E_n$$

Exam Review

- ***Mutually exclusive or disjoint events*** are two events for which

$$A \cap B = \emptyset$$

- A list of events $A_1, A_2, \ldots, A_n$ is said to be
  - composed of ***mutually exclusive events*** iff:

$$A_i \cap A_j = \begin{cases} A_i & \text{if } i = j \\ \emptyset & \text{otherwise} \end{cases}$$

  - *collectively* ***exhaustive*** iff: $A_1 \cup A_2 \cup \ldots \cup A_n = S$

Exam Review

- **Probability Axioms**
  - Let *S* be a sample space of a random experiment and *P(A)* be the probability of the event *A*
  - The probability function *P(.)* must satisfy the three following axioms:
  - **(A1)** For any event A, $P(A) \geq 0$

    (probabilities are nonnegative real numbers)
  - **(A2)** $P(S) = 1$

    (probability of a certain event, an event that must happen is equal 1)
  - **(A3)** $P(A \cup B) = P(A) + P(B)$, whenever A and B are mutually exclusive events, i.e., $A \cap B = \emptyset$

    (probability function must be additive)
  - **(A3')** For any countable sequence of events $A_1, A_2, \ldots, A_n \ldots$, that are mutually exclusive (that is $A_j \cap A_k = \emptyset$ whenever $j \neq k$)

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n)$$

Exam Review

- **(Ra)** For any event A, $P(\overline{A}) = 1 - P(A)$
- **(Rb)** If $\varnothing$ is the impossible event, then $P(\varnothing) = 0$
- **(Rc)** If A and B are any events, not necessarily mutually exclusive, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- **(Rd)**(generalization of Rc) If $A_1, A_2, \ldots, A_n$ are any events, then

$$P\left(\bigcup_{i=1}^{n} A_i\right) = P(A_1 \cup A_2 \cup \ldots \cup A_n) = \sum_i P(A_i) - \sum_{1 \le i < j \le n} P(A_i \cap A_j)$$

$$+ \sum_{1 \le i < j < k \le n} P(A_i \cap A_j \cap A_k) + \ldots + (-1)^{n-1} P(A_1 \cap A_2 \cap \ldots \cap A_n)$$

where the successive sums are over all possible events, pairs of events, triples of events, and so on.

(Can prove this relation by induction (see class web site))

Exam Review

- **Combinatorial Problems**
  - Permutations with replacement:
    - Ordered samples of size k, with replacement *P(n, k)*
  - Permutations without replacement
    - Ordered Samples of size k, without replacement

$$n(n-1)....(n-k+1) = \frac{n!}{(n-k)!} \qquad k = 1, 2, \ldots, n$$

  - Combinations
    - Unordered sample of size k, without replacement

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

- **Binomial Theorem**

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Exam Review

- **Conditional Probability** *of A given B (P(A|B))* defines the conditional probability of the event A given that the event B occurs and is given by:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

  if $P(B) \neq 0$ and is undefined otherwise.

- A rearrangement of the above definition gives the following ***multiplication rule (MR)***

$$P(A \cap B) = \begin{cases} P(B)P(A \mid B) & \text{if } P(B) \neq 0 \\ P(A)P(B \mid A) & \text{if } P(A) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

- Or:

$$P(A \cap B) = P(A)P(B|A) = P(B)P(A|B)$$

Exam Review

- **Theorem of Total Probability**
- Any event A can be partitioned into two disjoint subsets:

$$A = (A \cap B) \cup (A \cap \overline{B})$$

- Then:

$$P(A) = P(A \cap B) \cup P(A \cap \overline{B})$$
$$= P(A \mid B)P(B) + P(A \mid \overline{B})P(\overline{B})$$

- In general:

$$\boxed{P(A) = \sum_{i=1}^{n} P(A \mid B_i)P(B_i)}$$

- **Bayes Formula:**

$$\boxed{P(B_j \mid A) = \frac{P(B_j \cap A)}{P(A)} = \frac{P(A \mid B_j)P(B_j)}{\sum_i P(A \mid B_i)P(B_i)}}$$

Exam Review

- **Independence of Events:**
- Two events A and B are independent if and only if:

$$P(A|B)=P(A)$$

- Or events A and be are said to be independent if:

$$P(A \cap B) = P(A)P(B)$$

Exam Review

- **Reliability Applications:**
  - Recovery blocks
  - Series and parallel systems:
    - Series System: $R_s = P$ ("The system is functioning properly.")

$$= P(A_1 \cap A_2 \cdots \cap A_n)$$

$$= P(A_1)P(A_2)\cdots P(A_n)$$

$$= \prod_{i=1}^{n} R_i \qquad (2.1)$$

    - Parallel System:

$$R_p = 1 - F_p = 1 - \prod_{i=1}^{n}(1 - R_i)$$

    - In general: $$R_{sp} = \prod_{i=1}^{n}[1 - (1 - R_i)^{n_i}]$$
    - Bayes formula in example non series parallel systems

Exam Review

- **Bernoulli Trials**
  - The probability of obtaining exactly k successes in n trials is :

$$p(k) = \binom{n}{k} p^k q^{n-k} \qquad k = 0, 1, \ldots, n$$

- **NMR System:**

$$R_{m|n} = P(\text{"m or more components functioning properly"})$$

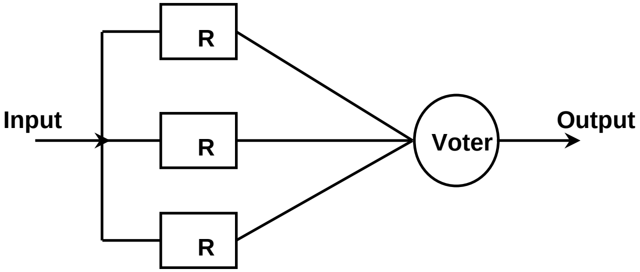
$$= P\left(\bigcup_{i=m}^{n} \{\text{"exactly i components functioning properly"}\}\right)$$

$$= \sum_{i=m}^{n} P(\text{"exactly i components functioning properly"})$$

$$= \sum_{i=m}^{n} p(i) = \sum_{i=m}^{n} \binom{n}{i} R^i (1-R)^{n-i}$$

- **TMR System:**

$$R_{TMR} = 3R^2 - 2R^3$$



Exam Review

- **Random Variables:**
  - A random variable X on a sample space S is a function $X: S \rightarrow \Re$ that assigns a real number X(s) to each sample point $s \in S$.
  - ***Discrete*** random variables: The random variables which are either finite or countable.
    - Bernoulli
    - Binomial
    - Poisson
    - Geometric
    - Modified Geometric
  - ***Continuous*** random variables: The random variables that take on a continuum of possible values.
    - Uniform
    - Normal
    - Exponential

Exam Review

- ***Cumulative distribution function* (cdf) (or *distribution function*)** $F(\cdot)$ of a random variable X is defined for any real number $b, -\infty < b < \infty$, by $F(b) = P\{X \leq b\}$
- $F(b)$ denotes the probability that the random variable X takes on a value that is less than or equal to b.

- Some properties of cdf F are:
  i. $F(b)$ is a non-decreasing function of b,
  ii. $\lim_{b \to +\infty} F(b) = F(\infty) = 1,$
  iii. $\lim_{b \to -\infty} F(b) = F(-\infty) = 0.$

- All probability questions about X can be answered in terms of cdf $F(\cdot)$. e.g.: $P\{a \leq X \leq b\} = F(b) - F(a) \qquad \text{for all } a < b$

Exam Review

- **Discrete Random Variables:**
  - ***Probability mass function (pmf)***:

$$p(a) = P\{X = a\}$$

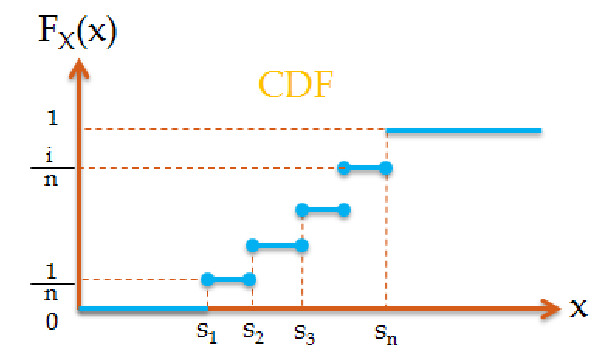
    - Properties:

$$\begin{cases} p(x_i) > 0, & i = 1,2,\ldots \\ p(x) = 0, & \text{for other values of x} \end{cases}$$

$$\sum_{i=1}^{\infty} p(x_i) = 1$$

  - **Cumulative distribution function (CDF)**:

$$F(a) = \sum_{all\ x_i \le a} p(x_i)$$



    - **A stair step function**

Exam Review

- **Continuous Random Variables:**
  - ***Probability distribution function (pdf)***:

$$P\{X \in B\} = \int_B f(x)\,dx$$

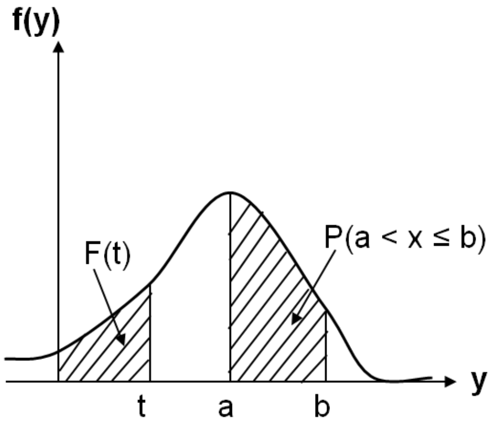
    - Properties:

$$1 = P\{X \in (-\infty, \infty)\} = \int_{-\infty}^{\infty} f(x)\,dx$$

    - All probability statements about X can be answered by $f(x)$:

$$P\{a \le X \le b\} = \int_a^b f(x)\,dx$$

$$P\{X = a\} = \int_a^a f(x)\,dx = 0$$

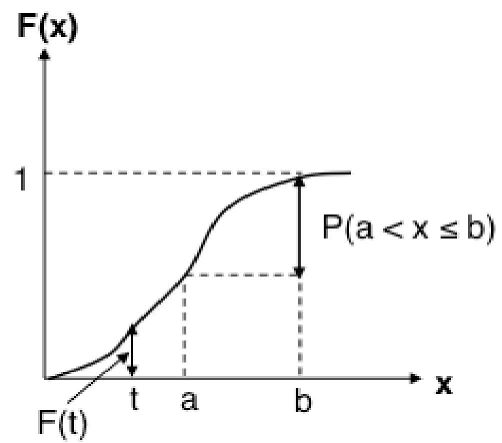


pdf

  - **Cumulative distribution function (CDF)**:

$$F_x(x) = P(X \le x) = \int_{-\infty}^{x} f_x(t)\,dt \;\;, -\infty < x < \infty$$

    - Properties: $\frac{d}{da}F(a) = f(a)$
    - **A continuous function**



CDF

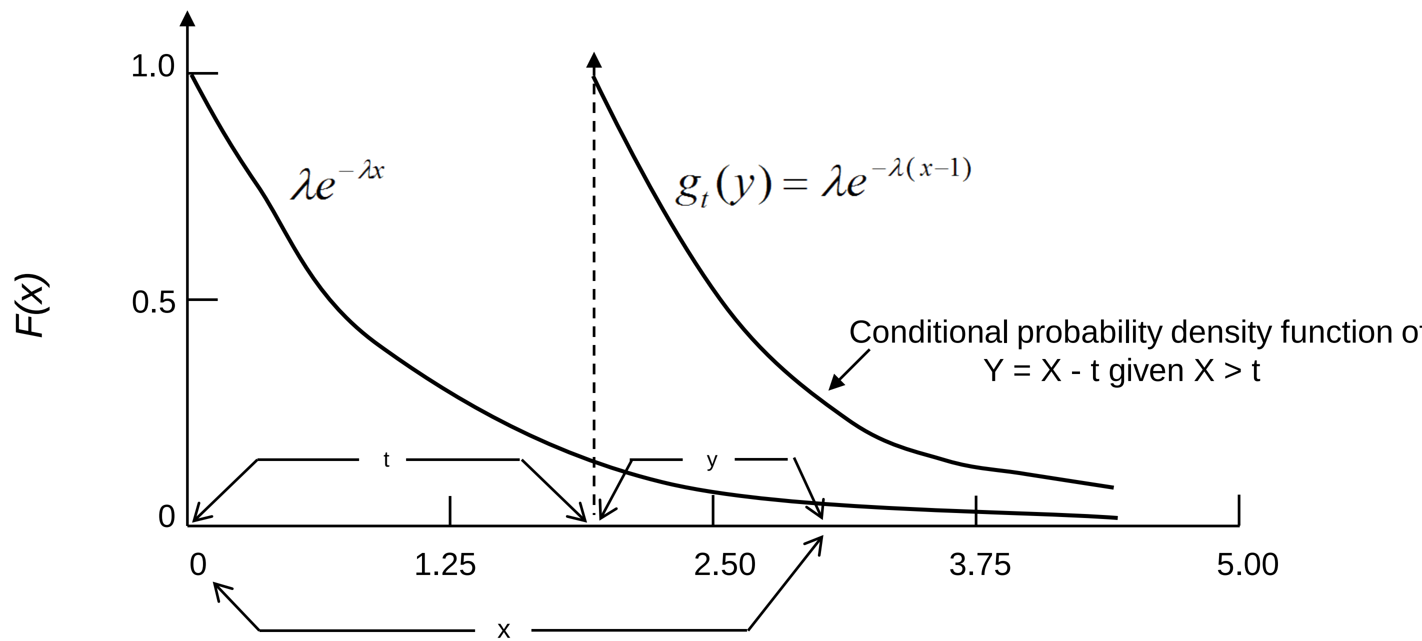
Exam Review

- **Summary of important distributions:**

Distribution	PDF or PMF	Mean	Variance
$Bernoulli(p)$	$\begin{cases} p, & \text{if } x = 1 \\ 1-p, & \text{if } x = 0. \end{cases}$	p	$p(1-p)$
$Binomial(n,p)$	$\binom{n}{k} p^k (1-p)^{n-k}$ for $0 \le k \le n$	np	npq
$Geometric(p)$	$p(1-p)^{k-1}$ for $k = 1, 2, \ldots$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
$Poisson(\lambda)$	$e^{-\lambda}\lambda^x/x!$ for $k = 1, 2, \ldots$	λ	λ
$Uniform(a,b)$	$\frac{1}{b-a}$ $\forall x \in (a,b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
$Gaussian(\mu,\sigma^2)$	$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	μ	σ^2
$Exponential(\lambda)$	$\lambda e^{-\lambda x}$ $x \ge 0, \lambda > 0$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$

Exam Review

- **Memory-less property of Exponential**



Exam Review

- **Expectation:**
  - The Discrete Case $E[X] = \sum_{x:p(x)>0} xp(x)$
  - The Continuous Case $E[X] = \int_{-\infty}^{\infty} xf(x)dx$

- **Expectation of function of a random variable**

$$E[g(X)] = \sum_{x:p(x)>0} g(x)p(x)$$

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)dx$$

  - **Corollary:**

$$E[aX + b] = aE[X] + b$$

Exam Review

- **Moments:**

$$E[Y]=E[\phi(X)]=\begin{cases}\sum_i \phi(x_i)\,p_X(x_i), & if\ X\ is\ discrete,\\ \int_{-\infty}^{\infty}\phi(x)\,f_X(x)dx, & if\ X\ is\ continuous,\end{cases}$$

$$Y=X^n \Rightarrow E[X^n],\ n\geq 1 \qquad \mu_k=E[(X-E[X])^k]$$

- **Variance:**

$$Var[X]=\mu^2=\sigma^2 x=\begin{cases}\sum_i (x_i-E[X])^2\,p(x_i) & if\ X\ is\ discrete\\ \int_{-\infty}^{\infty}(x-E[X])^2\,f(x)dx & if\ X\ is\ continuous\end{cases}$$

$$Var(X)=E[X^2]-(E[X])^2$$

  - **Corollary:**

$$Var[aX+b]=a^2Var(X)$$

Exam Review

- **Functions of random variables**
  - Examples shown in the class

- **Reliability function and Mean time to failure:**
  - Let *T* denote the time to failure or lifetime of a component in the system

$$R(t) = P\{T > t\} = 1 - F(t)$$

$$\boxed{E[T] = \int_0^\infty R(t)\,dt = MTTF}$$

  - If the component lifetime is exponentially distributed, then:

$$R(t) = e^{-\lambda t}$$

$$E[T] = \int_0^\infty e^{-\lambda t}\,dt = \frac{1}{\lambda}$$

$$Var[T] = \int_0^\infty 2te^{-\lambda t}\,dt - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

Exam Review

- **Joint distribution functions:**
  - For any two random variables *X* and *Y*, the ***joint cumulative probability distribution function*** of *X* and *Y* by:

$$F(a,b) = P\{X \le a, Y \le b\}, \; -\infty < a, b < \infty$$

  - **Discrete:**
    - The ***joint probability mass function*** of *X* and *Y*

$$p(x,y) = P\{X = x, Y = y\}$$

    - Marginal PMFs of X and Y:

$$p_X(x) = \sum_{y:p(x,y)>0} p(x,y)$$

$$p_Y(y) = \sum_{x:p(x,y)>0} p(x,y)$$

Exam Review

- **Joint distribution functions:**
  - **Continuous:**
    - The ***joint probability density function*** of *X* and *Y*:

$$P\{X \in A, Y \in B\} = \int_B \int_A f(x,y)\,dx\,dy$$

    - Marginal PDFs of X and Y:

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

    - Relation between joint CDF and PDF

$$F(a,b) = P(X \le a, Y \le b) = \int_{-\infty}^{a} \int_{-\infty}^{b} f(x,y)\,dy\,dx$$

Exam Review

- Function of two joint random variables

$$E[g(X,Y)] = \sum_y \sum_x g(x,y)\,p(x,y)$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\,f(x,y)\,dx\,dy$$

- For example, if $g(X,Y)=X+Y$, then, in the continuous case

$$E[X+Y] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x+y)f(x,y)\,dx\,dy$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xf(x,y)\,dx\,dy + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} yf(x,y)\,dx\,dy$$
$$= E[X] + E[Y]$$

Exam Review

- **Independent Random Variables:** Two random variables X and Y are said to be independent if:

$$F(x,y) = F_X(x)F_Y(y), \ -\infty < x < \infty, -\infty < y < \infty$$

- If X and Y are continuous:

$$f(x,y) = f_X(x)f_Y(y), \ -\infty < x < \infty, -\infty < y < \infty$$

- If X is discrete and Y is continuous:

$$P(X = x, Y \leq y) = p_X(x)f_Y(y), \text{ all } x \text{ and } y$$