Joint Distribution Functions

ECE 313
Probability with Engineering Applications
Lecture 15

Professor Ravi K. Iyer
Dept. of Electrical and Computer Engineering
University of Illinois at Urbana Champaign

Announcements

- **Homework 6 is due this Thursday, beginning of the class.**

- **Midterm next Tuesday, October 22**

  11:00am–12:20pm, in class
  - **All topics covered in Lectures 1 to 16**
  - **Homework 1-6, In-class projects 1-3, and Mini-Projects 1-2**

- **Review Session** on **Thursday, 5:00pm – 7:00pm, CSL 141**

- **Additional TA Office hours** on **Friday**, **2:00pm – 5pm, CSL 249.**

Today's Topics

- Quick Review: Expectation of Function of Random Variables
  - Examples
- Joint Distribution Functions
- Examples

Expectation of a Function of a Random Variable

- Given a random variable X and its probability distribution or its pmf/pdf
- We are interested in calculating not the expected value of X, but the expected value of some function of X, say, $g(X)$.
- **One way: since $g(X)$ is itself a random variable, it must have a probability distribution, which should be computable from a knowledge of the distribution of X. Once we have obtained the distribution of $g(X)$, we can then compute $E[g(X)]$ by the definition of the expectation.**
- Example 1: Suppose X has the following probability mass function:

$$p(0) = 0.2, \qquad p(1) = 0.5, \qquad p(2) = 0.3$$

- Calculate $E[X^2]$.
- Letting $Y=X^2$, we have that Y is a random variable that can take on one of the values, $0^2, 1^2, 2^2$ with respective probabilities

$$p_Y(0) = P\{Y = 0^2\} = 0.2$$
$$p_Y(1) = P\{Y = 1^2\} = 0.5$$
$$p_Y(2) = P\{Y = 2^2\} = 0.3$$

Hence,

$$E[X^2] = E[Y] = 0(0.2) + 1(0.5) + 4(0.3) = 1.7$$

Note that

$$1.7 = E[X^2] \neq E[X]^2 = 1.21$$

Expectation of a Function of a Random Variable (cont.)

- Proposition 2: (a) If X is a discrete random variable with probability mass function $p(x)$, then for any real-valued function g,

$$E[g(X)] = \sum_{x:p(x)>0} g(x)p(x)$$

- (b) if X is a continuous random variable with probability density function f(x), then for any real-valued function g:

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)dx$$

- Example 3, Applying the proposition to Example 1 yields

$$E[X^2] = 0^2(0.2) + (1^2)(0.5) + (2^2)(0.3) = 1.7$$

- Example 4, Applying the proposition to Example 2 yields

$$E[X^3] = \int_0^1 x^3 dx \qquad (\text{since } f(x) = 1,\ 0 < x < 1)$$

$$= \frac{1}{4}$$

Corollary

- If *a* and *b* are constants, then $E[aX+b]=aE[X]+b$
- **The discrete case:**

$$E[aX+b]=\sum_{x:p(x)>0}(ax+b)p(x)$$

$$=a\sum_{x:p(x)>0}xp(x)+b\sum_{x:p(x)>0}p(x)$$

$$=aE[X]+b$$

- **The continuous case:**

$$E[aX+b]=\int_{-\infty}^{\infty}(ax+b)f(x)dx$$

$$=a\int_{-\infty}^{\infty}xf(x)dx+b\int_{-\infty}^{\infty}f(x)dx$$

$$=aE[X]+b$$

Moments

- The expected value of a random variable *X, E[X],* is also referred to as the ***mean*** or the ***first moment*** of *X.*

- The quantity $E[X^n],\ n \geq 1$ is called the ***n*th moment** of *X.* We have:

$$E[X^n] = \begin{cases} \sum_{x:p(x)>0} x^n p(x), & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} x^n f(x)\,dx, & \text{if } X \text{ is continuous} \end{cases}$$

- Another quantity of interest is the variance of a random variable *X*, denoted by *Var(X),* which is defined by:

$$Var(X) = E[(X - E[X])^2]$$

Variance of a Random Variable

- Suppose that X is continuous with density *f*, let $E[X] = \mu$. Then,

$$
\begin{aligned}
Var(X) &= E[(X-\mu)^2] \\
&= E[X^2 - 2\mu X + \mu^2)] \\
&= \int_{-\infty}^{\infty} (x^2 - 2\mu x + \mu^2) f(x) dx \\
&= \int_{-\infty}^{\infty} x^2 f(x) dx - 2\mu \int_{-\infty}^{\infty} x f(x) dx + \mu^2 \int_{-\infty}^{\infty} f(x) dx \\
&= E[X^2] - 2\mu\mu + \mu^2 \\
&= E[X^2] - \mu^2
\end{aligned}
$$

- So we obtain the useful identity: $Var(X) = E[X^2] - (E[X])^2$

Example 1

- Let X be uniformly distributed in the unit interval [0, 1]. Consider the random variable $Y = g(X)$, where

$$g(x) = \begin{cases} 1, & if\ x \leq 1/3 \\ 2, & if\ x > 1/3 \end{cases}$$

- Find the expected value of Y by deriving its PMF. Verify the result using the expected value rule.

Example 1 (Cont'd)

Solution: The random variable $Y = g(X)$ is discrete and its PMF is given by

$$p_Y(1) = P(X \leq \frac{1}{3}) = \frac{1}{3}, \;\; p_Y(2) = 1 - p_Y(1) = \frac{2}{3}.$$

Thus,

$$E[Y] = \frac{1}{3} \times 1 + \frac{2}{3} \times 2 = \frac{5}{3}.$$

The same result is obtained using the expected value rule:

$$E[Y] = \int_0^1 g(x) f_X(x) dx = \int_0^{1/3} dx + \int_{1/3}^1 2dx = \frac{5}{3}.$$

Example 2

- Let *X* be a continuous random variable that is uniformly distributed on [−1, +1].
- Let *Y* = X^2. Calculate the mean and the variance of Y.

- We know that $E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x) dx$

- *Y* = g(X) = X^2 , so if *X* takes values between [-1,+1], *Y* takes values between [0,1], and we have:

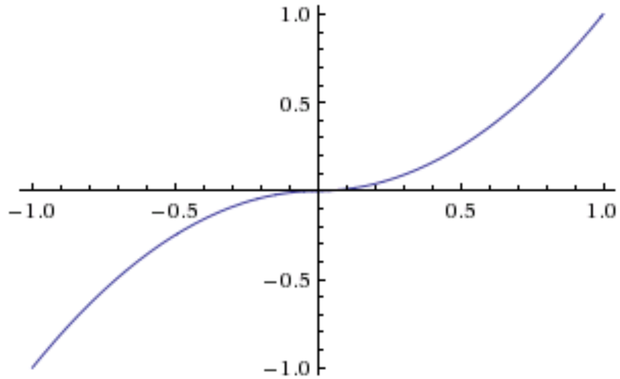
- E[Y] = E[X^2] = $\int_0^1 x^2 1 dx = \int_0^1 x^2 \, dx = \frac{x^3}{3} \Big|_0^1 = 1/3$

Example 2 (Cont'd)

- Let Z = g(X) where g(u) is defined as:

Find E[Z].

$$g(u) = \begin{cases} u^2, & u \geq 0 \\ -u^2, & u < 0 \end{cases}$$



- Since *X* takes values on [-1,1], we have:
- For $u \geq 0 \Rightarrow 0 < g(u) < 1$
- For $u < 0 \Rightarrow -1 < g(u) < 0$

$$E[Z] = \int_{-\infty}^{\infty} g(u) f(u) du = \int_0^1 u^2 .1\, du + \int_{-1}^0 (-u^2).1\, du$$

$$= \frac{u^3}{3}\Big|_0^1 - \frac{u^3}{3}\Big|_{-1}^0 = (\frac{1}{3} - 0) - (0 + \frac{1}{3}) = 0$$

Joint Distribution Functions

- We have concerned ourselves with the probability distribution of a single random variable
- Often interested in probability statements concerning two or more random variables
- Define, for any two random variables X and Y, the *joint cumulative probability distribution function* of X and Y by

$$F(a,b) = P\{X \le a, Y \le b\}, \; -\infty < a, b < \infty$$

- The distribution of X can be obtained from the joint distribution of X and Y as follows:

$$\begin{aligned} \mathrm{F_X}(a) &= P\{X \le a\} \\ &= P\{X \le a, Y < \infty\} \\ &= F(a, \infty) \end{aligned}$$

Joint Distribution Functions Cont'd

- Similarly, $F_Y(b) = P\{Y \le b\} = F(\infty, b)$ Where X and Y are both discrete random variables it is convenient to define the *joint probability mass function* of X and Y by $p(x, y) = P\{X = x, Y = y\}$

- Probability mass function of X $p_X(x) = \sum_{y:p(x,y)>0} p(x, y)$

$$p_Y(y) = \sum_{x:p(x,y)>0} p(x, y)$$

- We say that X and Y are *jointly continuous* defined for all real x and y

$$P\{X \in A, Y \in B\} = \int_B \int_A f(x, y)\,dx\,dy$$

Joint Distribution Functions Cont'd

- Called the *joint probability density function* of X and Y. The probability density of X

$$P\{X \in A\} = P\{X \in A, Y \in (-\infty, \infty)\}$$

$$= \int_{-\infty}^{\infty} \int_{A} f(x, y)dxdy$$

$$= \int_{A} f_X(x)dx$$

$f_X(x) = \int_{-\infty}^{\infty} f(x, y)dy$ is thus the probability density function of X

- Similarly the probability density function of Y is $f_Y(y) = \int_{-\infty}^{\infty} f(x, y)dx$ because

$$F(a,b) = P(X \leq a, Y \leq b) = \int_{-\infty}^{a} \int_{-\infty}^{b} f(x, y)dydx$$

Joint Distribution Functions Cont'd

- Proposition: if X and Y are random variables and g is a function of two variables, then

$$E[g(X,Y)] = \sum_y \sum_x g(x,y)p(x,y)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)f(x,y)dx\,dy$$

- For example, if $g(X,Y)=X+Y$, then, in the continuous case

$$E[X+Y] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x+y)f(x,y)dx\,dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xf(x,y)dx\,dy + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} yf(x,y)dx\,dy$$

$$= E[X]+E[Y]$$

Joint Distribution Functions Cont'd

- Where the first integral is evaluated by using the foregoing Proposition with $g(x,y)=x$ and the second with $g(x,y)=y$
- In the discrete case $E[aX+bY]=aE[X]+bE[Y]$
- Joint probability distributions may also be defined for n random variables. If $X_1, X_2, \ldots, X_n$ are n random variables, then for any n constants $a_1, a_2, \ldots, a_n$

$$E[a_1X_1+a_2X_2+\ldots a_nX_n]=a_1E[X_1]+a_2E[X_2]+\ldots+a_nE[X_n]$$

Example 1

- A batch of 1M RAM chips are purchases from two different semiconductor houses. Let *X* and *Y* denote the times to failure of the chips purchased from the two suppliers. The joint probability density of *X* and *Y* is estimated by:

$$f(x,y) = \begin{cases} \lambda\mu e^{-(\lambda x+\mu y)}, & x > 0, y > 0 \\ 0, & otherwise \end{cases}$$

- Assume $\lambda = 10^{-5}$ per hour and $\mu = 10^{-6}$ per hour.
- Determine the probability that time to failure is greater for chips characterized by *X* than it is for chips characterized by *Y.*

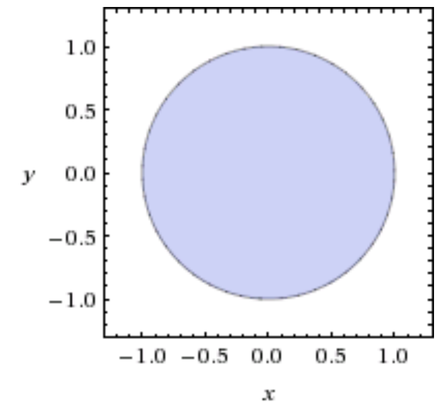
Example 1 (Cont'd)

$$
\begin{aligned}
P(X \geq Y) &= \int_{x=0}^{\infty} \int_{y=0}^{x} f(x,y)\; dy\; dx \\
&= \int_{x=0}^{\infty} \left[\int_{y=0}^{x} \mu e^{-\mu y}\; dy \right] \lambda\, e^{-\lambda x}\; dx \\
&= \int_{0}^{\infty} \left[1 - e^{-\mu x} \right] \lambda e^{-\lambda x}\; dx \\
&= 1 \;-\; \lambda \, \frac{e^{-(\lambda+\mu)x}}{-(\lambda+\mu)} \Big|_{0}^{\infty} \;=\; 1 \;-\; \frac{\lambda}{\lambda+\mu} \\
&= \frac{\mu}{\lambda+\mu} \\
&= \frac{10^{-6}}{10^{-5} \;+\; 10^{-6}} \;=\; \frac{1}{11} \;=\; 0.0909.
\end{aligned}
$$

Example 2

- Let X and Y have joint pdf

$$f(x,y) = \begin{cases} \frac{1}{\pi}, & x^2 + y^2 \le 1, \\ 0, & \text{otherwise.} \end{cases}$$



Determine the marginal pdfs of *X* and *Y*. Are X and Y independent?

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f(x,y)\ dy \\ &= \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi}\ dy \\ &= \frac{2}{\pi}\sqrt{1-x^2}, \qquad -1 < x < 1. \end{aligned}$$

Example 2 (Cont'd)

- Similarly,

$$f_Y(y) \;=\; \frac{2}{\pi}\,\sqrt{1-y^2}, \qquad -1 < y < 1.$$

- So clearly,

$$f(x,y) \;\neq f_X(x)\; f_Y(y) \Rightarrow X \text{ and } Y \text{ are not independent.}$$

Example 3

Suppose that the joint probability mass function of X and Y is

$$P(X = i, Y = j) = \binom{j}{i} e^{-2\lambda} \lambda^j / j!, \quad 0 \leq i \leq j$$

(a) Find the probability mass function of Y.
(b) Find the probability mass function of X.
(c) Find the probability mass function of $Y - X$.

Example 3 (Cont'd)

a) Marginal PDF of *Y*:

$$P(Y=j) = \sum_{i=0}^{j} \binom{j}{i} e^{-2\lambda}\lambda^j/j!$$
$$= e^{-2\lambda}\frac{\lambda^j}{j!}\sum_{i=0}^{j}\binom{j}{i}1^i 1^{j-i}$$
$$= e^{-2\lambda}\frac{(2\lambda)^j}{j!}$$

b) Marginal PDF of *X:*

$$P(X=i) = \sum_{j=i}^{\infty} \binom{j}{i} e^{-2\lambda}\lambda^j/j!$$
$$= \frac{1}{i!}e^{-2\lambda}\sum_{j=i}^{\infty}\frac{\lambda^j}{(j-i)!}$$
$$= \frac{\lambda^i}{i!}e^{-2\lambda}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}$$
$$= e^{-\lambda}\frac{\lambda^i}{i!}.$$

Example 3 (Cont'd)

c) We first calculate the joint density function of *X* and *Y-X*

$$
\begin{aligned}
P(X=i, Y-X=k) &= P(X=i, Y=k+i) \\
&= \binom{k+i}{i} e^{-2\lambda} \frac{\lambda^{k+i}}{(k+i)!} \\
&= e^{-2\lambda} \frac{\lambda^k}{k!} \frac{\lambda^i}{i!}.
\end{aligned}
$$

- Then summing up with respect to *i*, we get the marginal distribution of *Y – X*, which is for *k:*

$$
\begin{aligned}
P(Y-X=k) &= \sum_{i=0}^{\infty} P(X=i, Y-X=k) \\
&= e^{-2\lambda} \frac{\lambda^k}{k!} \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} \\
&= e^{-2\lambda} \frac{\lambda^k}{k!} e^{-\lambda} \\
&= e^{-\lambda} \frac{\lambda^k}{k!}.
\end{aligned}
$$