Continuous Random Variables and Distributions

ECE 313

Probability with Engineering Applications

Lecture 10

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Today's Topics

- Probability Density Function
- Relation between CDF and PDF

- Normal or Gaussian Distribution
- Exponential Distribution

Probability Density Function

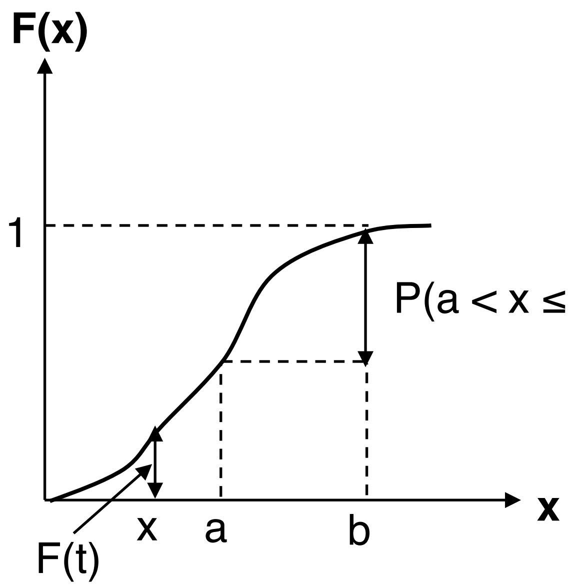
- For a continuous random variable, *X, f(x) = dF(x)/dx* is called the probability density function (pdf or density function) of *X*.
- The pdf enables us to obtain the CDF by integrating under the pdf:

$$F_x(x) = P(X \le x) = \int_{-\infty}^{x} f_x(t)dt \qquad -\infty < x < \infty$$

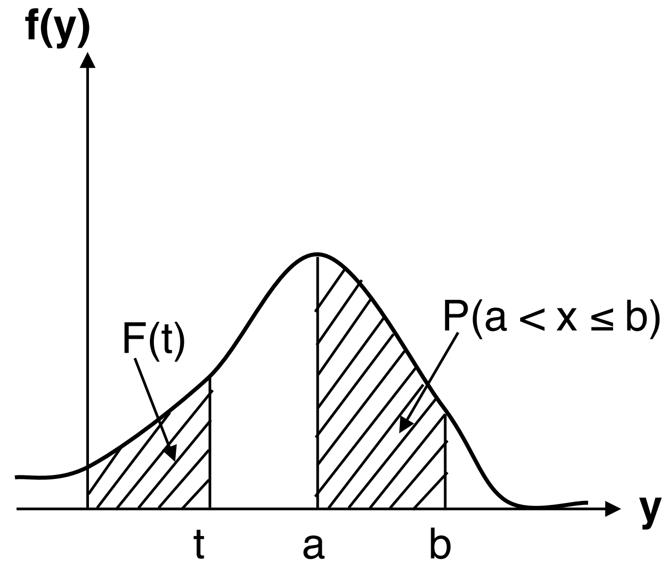
- We can also obtain other probabilities of interest such as:

$$\begin{aligned} P(X \in (a,b]) &= P(a < X \le b) \\ &= P(x \le b) - P(X \le a) \\ &= \int_{-\infty}^{b} f_x(t)dt - \int_{-\infty}^{a} f_x(t)dt \\ &= \int_{a}^{b} f_x(t)dt \end{aligned}$$

Relation Between CDf and pdf



CDF



pdf

Probability Density Function (cont.)

- Unlike the pmf, the values of the pdf are not probabilities and it is acceptable if $f(x) > 1$ at point x.

- As is the case for the CDF of a discrete random variable, the CDF continuous random variable, $F(t)$ satisfies the following properties:

  (F1) $0 \leq F(t) \leq 1$ for $-\infty < t < \infty$,

  (F2) $F(t)$ is a monotone non-decreasing function of t

  (F3) $\lim_{t \to -\infty} F(t) = 0$ and $\lim_{t \to \infty} F(t) = 1$

- Unlike the CDF of a discrete random variable, the CDF of a continuous random variable does not have any jumps.

Probability Density Function (cont.)

- The probability associated with event $[X = c] = \{s|X(s) = c\} = 0$:

(F4´) $$P(X = c) = P(c \le X \le c) = \int_c^c f_x(y)dy = 0$$

- Because $P(X = c) = 0$, we have:

$$P(a \le X \le b) = P(a < X \le b) = P(a \le X < b)$$
$$= P(a < X < b)$$
$$= \int_a^b f_x(x)dx$$
$$= F_x(b) - F_x(a)$$

- So for the probability density function we have these properties:

$$f_x(x) \ge 0 \quad for\ all\ x \ge 0, \qquad \int_{-\infty}^{+\infty} f_x(x)dx = 1$$

Example

- The time (measured in years), X, required to complete a software project has a pdf of the form:

$$f_x(x) = \begin{cases} kx(1-x), & 0 \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

- Since f_x satisfies property $f_x(x) \ge 0$, we know $k \ge 0$.
- In order for f_x to be a pdf, it must also satisfy (intergral over its full range = 1):

$$\int_0^1 kx(1-x)dx = 1$$

$$k\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Bigg|_0^1 = 1 \quad \text{therefore: } k = 6$$

- The probability that the project will be completed in less than 4 months is given by:

$$P(X < 4/12) = F_x(1/3) = \int_0^{1/3} f_x(x)dx = 7/27$$

- About a 26 percent chance

Normal or Gaussian Distribution

Normal or Gaussian Distribution

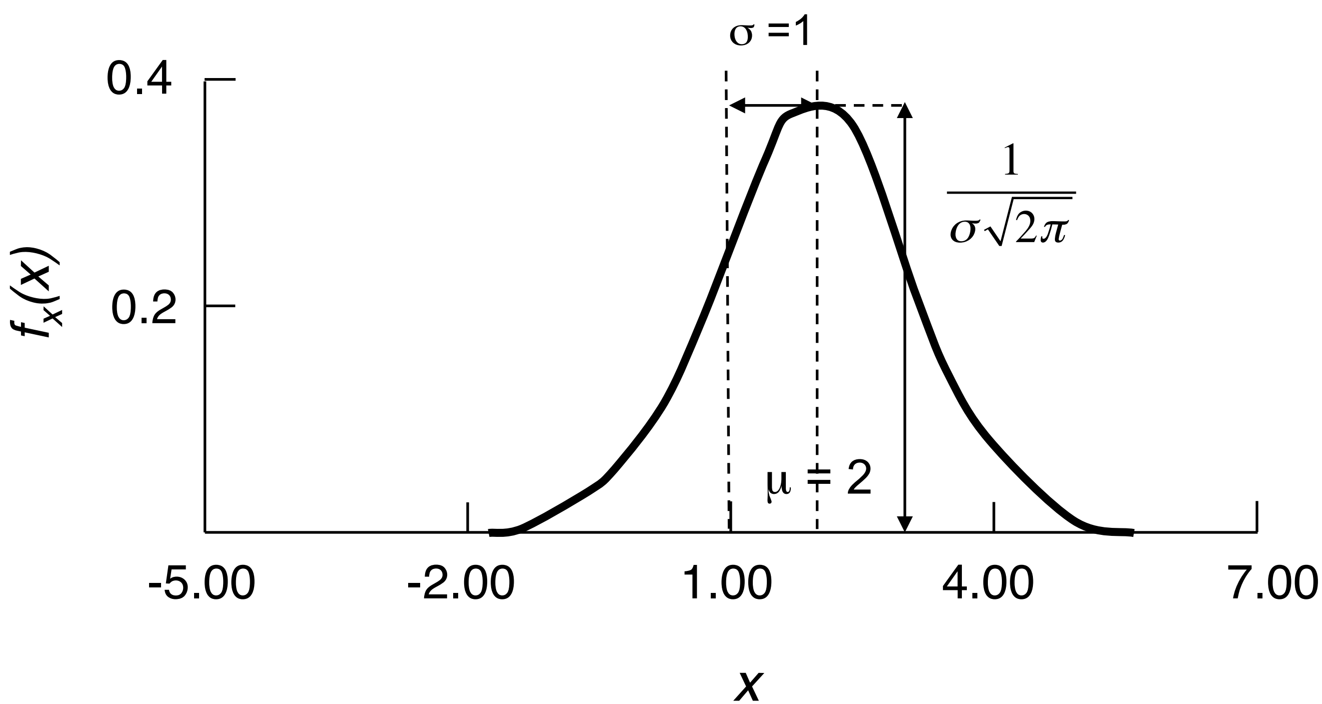
- Extremely important in statistical application because of the central limit theorem:
  - Under very general assumptions, the mean of a sample of n mutually independent random variables is normally distributed in the limit $n \to \infty$.
- Errors of measurement often possess this distribution.
- During the wear-out phase, component lifetime follows a normal distribution.
- The normal density is given by:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right], \qquad -\infty < x < \infty$$

where $-\infty < \mu < \infty$ and $\sigma > 0$ are two parameters of the distribution.

Normal or Gaussian Distribution (cont.)

- Normal density with parameters μ =2 and σ =1



Normal or Gaussian Distribution (cont.)

- The distribution function *F(x)* has no closed form, so between every pair of limits *a* and *b*, probabilities relating to normal distributions are usually obtained numerically and recorded in special tables.

- These tables pertain to the **standard normal distribution** [$Z \sim N(0,1)$] --- a normal distribution with parameters $\mu = 0$, $\sigma = 1$ --- and their entries are the values of:

$$F_Z(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2} dt$$

Normal or Gaussian Distribution (cont.)

- Since the standard normal density is clearly symmetric, it follows that for $z > 0$:

$$
\begin{aligned}
F_Z(-z) &= \int_{-\infty}^{-z} f_Z(t)dt \\
&= \int_{z}^{\infty} f_Z(-t)dt \\
&= \int_{z}^{\infty} f_Z(t)dt \\
&= \int_{-\infty}^{\infty} f_Z(t)dt - \int_{-\infty}^{z} f_Z(t)dt \\
&= 1 - F_Z(z)
\end{aligned}
$$

- The tabulations of the normal distribution are made only for $z \geq 0$ To find $P(a \leq Z \leq b)$, use $F(b) - F(a)$.

Normal or Gaussian Distribution (cont.)

- The CDF of the N(0,1) distribution ($F_Z(z)$) is denoted in the tables by Φ , and its complementary CDF is denoted by Q, so:

$$Q(u) = 1 - \Phi(u) = \Phi(-u)$$

Table 6.1: Φ function, the area under the standard normal pdf to the left of x.

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

Table 6.2: Q function, the area under the standard normal pdf to the right of x.

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641
0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010

Normal or Gaussian Distribution (cont.)

- For a particular value, *x*, of a normal random variable *X*, the corresponding value of the standardized variable *Z* is given by:

$$Z = (X - \mu) / \sigma$$

- The distribution function of *X* can be found by using:

$$F_Z(z) = P(Z \le z)$$
$$= P(\frac{X - \mu}{\sigma} \le z)$$
$$= P(X \le \mu + z\sigma)$$
$$= F_X(\mu + z\sigma)$$

alternatively:

$$F_X(x) = F_Z(\frac{x - \mu}{\sigma})$$

- Similarly, if *X* is normally distributed with parameters *μ* and *σ2* then $Z = \alpha X + \beta$ is normally distributed with parameters $\alpha\mu + \beta$ and $\alpha^2\sigma^2$.

Normal or Gaussian Distribution Example 1

- An analog signal received at a detector (measured in microvolts) may be modeled as a Gaussian random variable N(200, 256) at a fixed point in time. What is the probability that the signal will exceed 240 microvolts? What is the probability that the signal is larger than 240 microvolts, given that it is larger than 210 microvolts?

$$
\begin{aligned}
P(X > 240) &= 1 - P(X \leq 240) \\
&= 1 - F_Z\left(\frac{240 - 200}{16}\right) \\
&= 1 - F_Z(2.5) \\
&\approx 0.00621
\end{aligned}
$$

Normal or Gaussian Distribution Example 1 (cont.)

- Next:

$$P(X \geq 240 \mid X \geq 210) = \frac{P(X) \geq 240)}{P(X \geq 210)}$$

$$= \frac{1 - F_Z\left(\dfrac{240 - 200}{16}\right)}{1 - F_Z\left(\dfrac{210 - 200}{16}\right)}$$

$$= \frac{0.00621}{0.26599}$$

$$\approx 0.02335$$

Normal or Gaussian Distribution (cont.)

- It is of interest to define a ***truncated normal density***:

$$f(x) = \begin{cases} 0, & x < 0 \\ \dfrac{1}{\alpha\sigma\sqrt{2}} \exp\left[\dfrac{-(x-\mu)^2}{2\sigma^2}\right], & x \geq 0 \end{cases}$$

- Where:

$$\alpha = \int_0^\infty \frac{1}{\sigma\sqrt{2\pi}} \exp\left[\frac{-(t-\mu)^2}{2\sigma^2}\right] dt$$

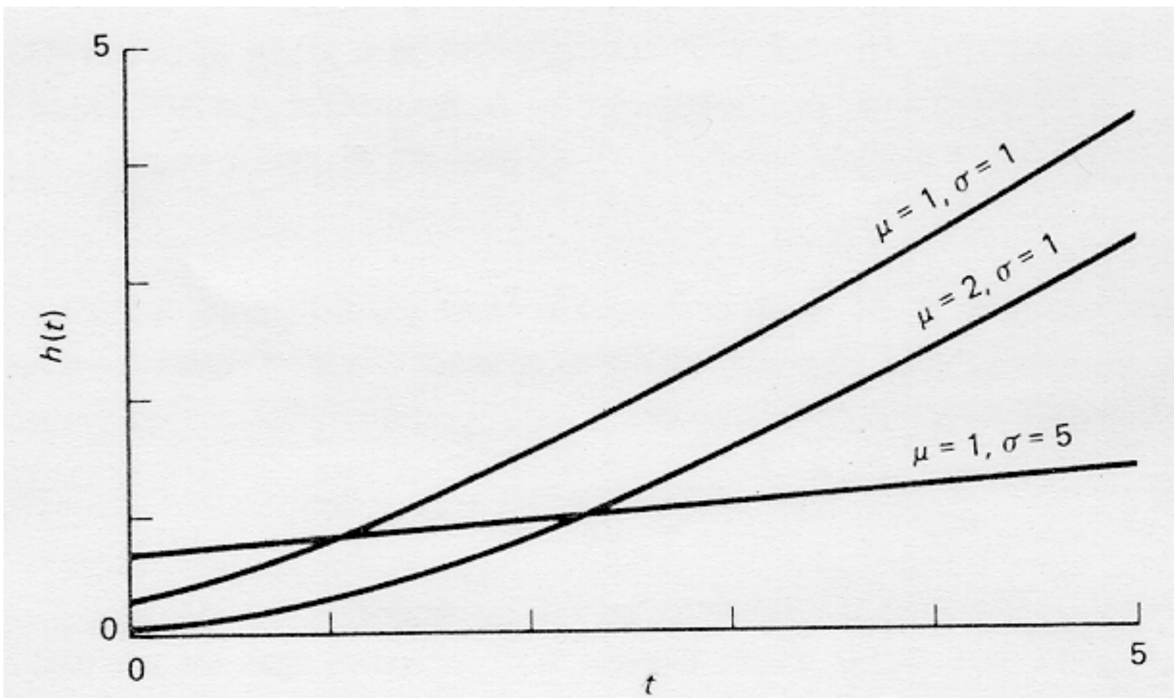
Normal or Gaussian Distribution (cont.)

- The introduction of α insures that $\int_{-\infty}^{\infty} f(t)dt = 1$

  so that *f* is the density of a nonnegative random variable.
- For $\mu > 3\sigma$, the value of α is close to 1, and for most practical purposes it may be omitted, so that the truncated normal density reduces to the usual normal density.

Failure Rate of the Normal Distribution

- The normal distribution is IFR, which implies that it can be used to model the behavior of components during the wear-out phase.



Normal or Gaussian Distribution Example 2

- Assuming that the life of a given subsystem, in the wear-out phase, is normally distributed with μ = 10,000 hours and σ = 1,000 hours, determine the reliability for an operating time of 500 hours given that
  - (a) The age of the component is 9,000 hours,
  - (b) The age of the component is 11,000.

- The required quantity under (a) is $R_{9,000}(500)$ and under (b) is $R_{11,000}(500)$.

Normal or Gaussian Distribution Example 2 (cont.)

- Note that with the usual exponential assumption these two quantities will be identical.
- But in the present case:

$$R_{9,000}(500) = \frac{R(9,500)}{R(9,000)}$$

$$= \frac{\int_{9,500}^{\infty} f(t)dt}{\int_{9,000}^{\infty} f(t)dt}$$

Normal or Gaussian Distribution Example 2 (cont.)

Noting that $\mu - 0.5\sigma = 9{,}500$ and $\mu - \sigma = 9{,}000$, we have:

$$R_{9,000}(500) = \frac{\int_{\mu-0.5\sigma}^{\infty} f(t)dt}{\int_{\mu-\sigma}^{\infty} f(t)dt}$$

$$= \frac{1 - F_X(\mu - 0.5\sigma)}{1 - F_X(\mu - \sigma)} = \frac{1 - F_Z(-0.5)}{1 - F_Z(-1)} = \frac{F_Z(0.5)}{F_Z(1)}$$

$$= \frac{0.6915}{0.8413} \quad \text{(from tables)}$$

$$= 0.8219$$

Normal or Gaussian Distribution Example 2 (cont.)

Similarly, since $\mu + 1.5\sigma = 11{,}500$ and $\mu + \sigma = 11{,}000$, we have:

$$R_{11,000}(500) = \frac{1 - F_X(\mu + 1.5\sigma)}{1 - F_X(\mu + \sigma)}$$

$$= \frac{0.0668}{0.1587} \qquad \text{(from tables)}$$

$$= 0.4209$$

Thus, unlike the exponential assumption, $R_{11,000}(500) < R_{9,000}(500)$;

that is, the subsystem has aged.

It can be shown that the normal distribution is a good approximation to the (discrete) binomial distribution for large n, provided p is not close to 0 or 1. The corresponding parameters are $\mu = np$ and $\sigma^2 = np(1 - p)$.

Exponential Distribution

The Exponential Distribution

- The exponential distribution occurs in applications such as reliability theory and queuing theory. Reasons for its use include:
  - Its memoryless (Markov) property
  - Its relation to the (discrete) Poisson distribution

- The following random variables will often be modeled as exponential:
  - Time between two successive job arrivals to a computing center
  - Service time at a server in a queuing network
  - Time to failure (lifetime) of a component
  - Time required to repair a component that has malfunctioned

The Exponential Distribution Function

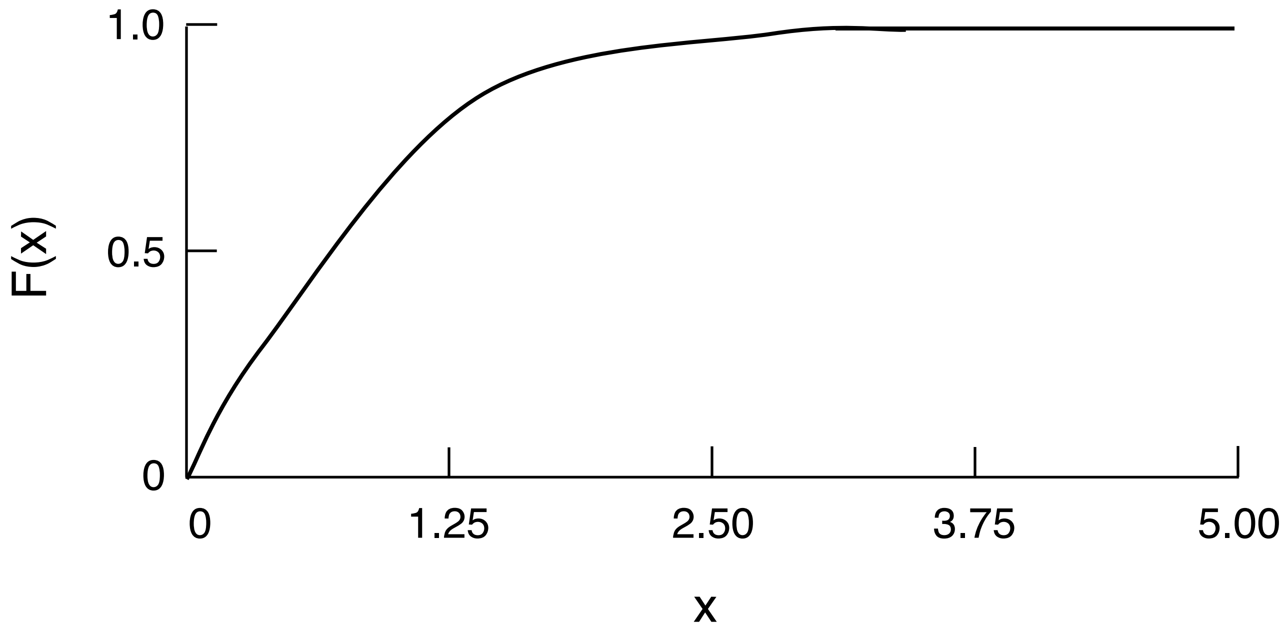
- The exponential distribution function is given by:

$$F(x) = \begin{cases} 1 - e^{-\lambda x}, & \text{if } 0 \le x < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

- If a random variable *X* possesses CDF given in the above equation, we use the notation *X* ~EXP(λ), for brevity. The pdf of *X* is given by:

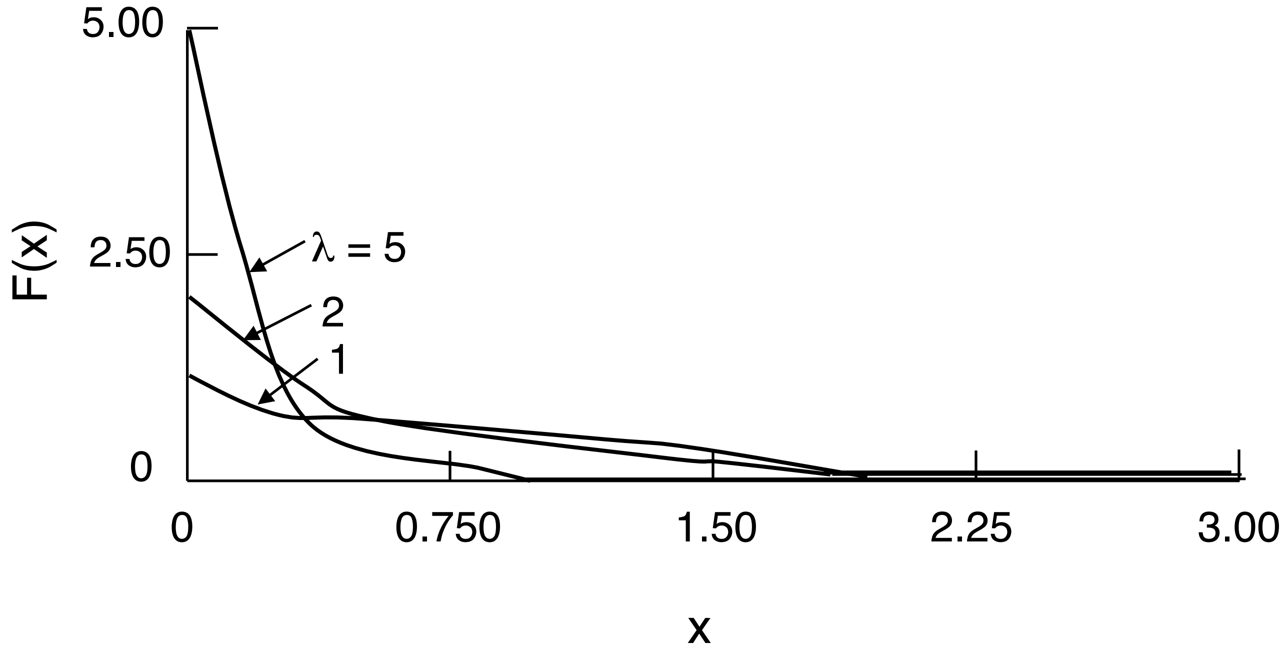
$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x > 0. \\ 0, & \text{otherwise} \end{cases}$$

The CDF of an Exponentially Distributed Random Variable



- The CDF of an exponentially distributed random variable with parameter $\lambda = 1$

Exponential pdf



Specifying the pdf

- While specifying the pdf, we usually state only the nonzero part.
- It is understood that the pdf is zero over any unspecified region.
- Since $\lim_{x \to +\infty} \mathrm{F(x)} = 1$, the total area under the exponential pdf is unity.
- Also:

$$P(X \geq t) = \int_t^\infty f(x)dx = e^{-\lambda t}$$

and

$$P(a \leq X \leq b) = F(b) - F(a) = e^{-\lambda a} - e^{-\lambda b}$$