Exponential Distribution and Related Properties:

If X is the time to failure lifetime of system and is exponentially distributed with parameter λ, then the PDF of X, the failure density function, is given by:

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x > 0. \\ 0, & \text{otherwise} \end{cases}$$

The CDF of X is given by:

$$F(x) = \begin{cases} 1 - e^{-\lambda x}, & \text{if } 0 \le x < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

The probability that the component survives until some time *t* is called the reliability *R(t)* of the component:

$$R(t) = P(X > t) = 1 - F(t)$$

The instantaneous failure rate *h(t)* at time *t* is defined to be:

$$h(t) = \lim_{x \to 0} \frac{1}{x} \frac{F(t+x) - F(t)}{R(t)} = \lim_{x \to 0} \frac{R(t) - R(t+x)}{xR(t)}$$

$$h(t) = \frac{f(t)}{R(t)}$$

An exponential distribution is characterized by a constant instantaneous hazard function or failure rate:

$$h(t) = \frac{f(t)}{R(t)} = \frac{\lambda e^{-\lambda t}}{e^{-\lambda t}} = \lambda$$

$R(t) = \exp\left[-\int_0^t h(x)dx\right]$: A useful theoretical representation of reliability as a function of the hazard rate.

Density Function	Hazard Rate
f(t) for $0 < t \le \infty$ Density function is defined for all positive time	z(x) for $0 < t \le \infty$ Hazard rate is defined for all positive time
$f(t) \ge 0$ f(t) is never negative	$z(t) \ge 0$ z(t) is never negative
$\int_0^\infty f(t)dt = 1$ Probability of sample space is unity	$\int_0^\infty z(t)dt = \infty$ Equivalent to condition on f(t)

Phase-type Exponential Distributions: If we have a process that is divided into r sequential phases, in which time that the process spends in each phase is:

- Independent
- Exponentially distributed

Four special types of phase-type exponential distributions:

1) **Hypoexponential Distribution:**
   - Exponential distributions at each phase have different λ
   - For example, a two-stage hypoexponential random variable, X, with parameters λ_1 and λ_2 ($\lambda_1 \neq \lambda_2$), is denoted by X~HYPO(λ_1, λ_2), has the following density and distribution functions and hazard rate:

$$f(t) = \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1}(e^{-\lambda_1 t} - e^{-\lambda_2 t}), \quad t > 0$$

$$F(t) = 1 - \frac{\lambda_2}{\lambda_2 - \lambda_1} e^{-\lambda_1 t} + \frac{\lambda_1}{\lambda_2 - \lambda_1} e^{-\lambda_2 t}, \quad t \geq 0$$

$$h(t) = \frac{\lambda_1 \lambda_2 (e^{-\lambda_1 t} - e^{-\lambda_2 t)}}{\lambda_2 e^{-\lambda_1 t} - \lambda_1 e^{-\lambda_2 t}}$$

2) **r-stage Erlang Distribution:**
   - Exponential distributions in each phase are identical (with same λ)
   - The number of phases (α) is an integer
   - An r-stage Erlang random variable, has the following density and distribution functions and hazard rate:

$$f(t) = \frac{\lambda^r t^{r-1} e^{-\lambda t}}{(r-1)!}, \quad t > 0, \lambda > 0, r = 1,2,...$$

$$F(t) = 1 - \sum_{k=0}^{r-1} \frac{(\lambda t)^k}{k!} e^{-\lambda t}, \quad t \leq 0, \lambda > 0, r = 1,2,...$$

$$h(t) = \frac{\lambda^r t^{r-1}}{(r-1)! \sum_{k=0}^{r-1} \frac{(\lambda t)^k}{k!}}, \quad t > 0, \lambda > 0, r = 1,2,...$$

3) **Gamma Distribution**
   - Is a R-stage Erlang
   - But the number of phases (α) is not an integer

4) **Hyperexponential Distribution:**
   - A mixture of different exponential distributions

Erlang and Hypoexponential Examples: Consider a dual redundant system composed of two identical components, where the secondary component acts as a backup of the primary one and will be powered on only after the primary component fails. A detector circuit checks the output of primary component in order to identify its failure and a switch is used to configure and power on the secondary component. Let the lifetime of the two components be two independent random variables X_1 and X_2, which are exponentially distributed with parameter λ.

Part a) Assume that the detection and switching circuits are perfect. What is the distribution of time to failure of the whole system? Derive the reliability function for the system.

Solution: Total lifetime of the system can be modeled by a **2-stage Erlang random variable** with the following density and reliability function:

$$f_z(t) = \int_0^t \lambda e^{-\lambda x} \lambda e^{-\lambda(t-x)} dx = \lambda^2 e^{-\lambda t} \int_0^t dx = \lambda^2 e^{-\lambda t}, t > 0$$

$$F_z(t) = \int_0^t f_z(z)\, dz = 1 - (1 + \lambda t)e^{-\lambda t}$$

$$R(t) = 1 - F(t) = F_z(t) = (1 + \lambda t)e^{-\lambda t}, t \geq 0$$

You can only use the r-stage Erlang formulas in the previous page (with r = 2), to derive these functions.

Part b) Assume X_1 and X_2 are exponentially distributed with parameters $\lambda 1$ and $\lambda 2$ ($\lambda_1 \neq \lambda_2$), what would be the distribution of time to failure of the system?

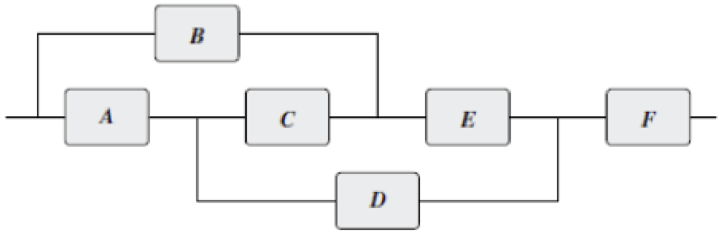
Solution: Since here the exponential distributions at each phase have different parameters $\lambda 1$ and $\lambda 2$ ($\lambda_1 \neq \lambda_2$), the total lifetime of the system can be modeled by a **2-stage Hypoexponential random variable** with the failure density function:

$$f(t) = \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1}(e^{-\lambda_1 t} - e^{-\lambda_2 t}), \quad t > 0$$

And the reliability of the system would be:

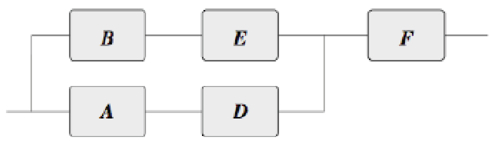
$$F(t) = \frac{\lambda_2}{\lambda_2 - \lambda_1} e^{-\lambda_1 t} + \frac{\lambda_1}{\lambda_2 - \lambda_1} e^{-\lambda_2 t}, \quad t \geq 0$$

Non-Series-Parallel System Reliability Example: Consider the non-series-parallel system of six independent components shown in the following figure. Assume that the reliability of each component is equal to R. Determine an expression for the system reliability as a function of component reliabilities.

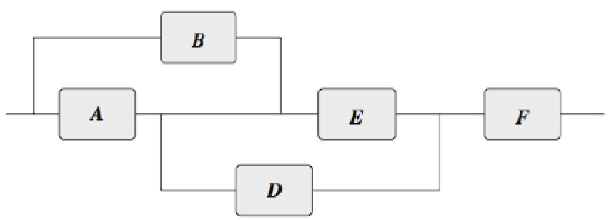


Solution: In order to solve this kind of problems, we need to divide the system around a node into two separate sub-systems that their reliabilities can be calculated as series-parallel systems. The process of dividing is repeated until the resulting subsystems are of the series/parallel type. Then we should condition the reliability of the system on whether or not that module is functional, and use the Total Probability formula.

Here, we pick node C to divide the system as follows:



(a) C not working



(b) C working

The part (a) is already of the series/parallel system, whereas part b needs further dividing around node E. Note that part b should not be viewed as a parallel connection of A and B, connected serially to D and E in parallel; such a diagram will have the path $BCDF$, which is not a valid path in the original system.

$R_{\text{system}} = R_c \cdot \text{Prob}\{\text{System works} \mid C \text{ is fault-free}\} + (1 - Rc) \cdot \text{Prob}\{\text{System works} \mid C \text{ is faulty}\}$

If C is not working, Prob{System works | C is faulty} $= R_F\,[1-(1-R_A R_D)(1-R_B R_E)]$

If C is working, Prob{System works|C is fault-free} $= R_E R_F\,[1-(1-R_A)(1-R_B)]+(1-R_E)R_A R_D R_F$

$R\text{system}=R_C[R_E R_F\,(R_A + R_B - R_A R_B) + (1-R_E)\,R_A R_D R_F] + (1-RC)[R_F\,(R_A R_D + R_B R_E - R_A R_D R_B R_E)]$

If $R_A = R_B = R_C = R_D = R_E = R_F = R$, then $R_{\text{system}} = R^3.\,(R^3 - 3R^2 + R + 2)$

<u>**See the solution to problem 3 of the midterm exam as a simpler example.**</u>

Covariance Example: **Read problem 2 of Quiz 2.** Further explanation is provided here:

Suppose n fair dice are independently rolled. Let:

$$X_k = \begin{cases} 1 & if\ 1\ shows\ on\ the\ k^{th}\ die \\ 0 & else \end{cases} \qquad Y_k = \begin{cases} 1 & if\ 2\ shows\ on\ the\ k^{th}\ die \\ 0 & else \end{cases}$$

Let $X = \sum_{k=1}^{n} X_k$, which is the number of one's showing, and $Y = \sum_{k=1}^{n} Y_k$, which is the number of two's showing. Note that if a histogram is made recording the number of occurrences of each of the six numbers, then X and Y are the heights of the first two entries in the histogram.

a) Find $E[X]$ and $Var(X)$.
b) Find $Cov(X_i, Y_j)$ if $1 \le i \le n$ and $1 \le j \le n$ (**Hint:** Does it make a difference if i = j?)

Solution: There are four important properties to use for solving this problem:

For any series of random variables $X_1, X_2, X_3, \ldots, X_n$, we have:

$$E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E(X_i)$$

If variables $X_1, X_2, X_3, \ldots, X_n$, **are independent**

$$Var\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} Var(X_i)$$

In order to calculate the Cov(X,Y) using the following formula:

$$Cov(X,Y) = E[XY] - E[X]E[Y]$$

We can get the E[XY] for any two continuous random variables X and Y from their joint distribution function f(x,y) as follows:

$$E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x,y)dy\,dx$$

We can get the E[XY] for any two discrete random variables X and Y from their joint probability mass function p(x,y) as follows:

$$E[XY] = \sum xy\, p(x,y)$$

Part a) we have:

$$E(X) = E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E(X_i) = nE(X_k) = \frac{n}{6}$$

And since different rolls of a dice X_k are independent from each other, we have:

$$Var(X) = Var\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} Var(X_i) = nVar(X_k) = \frac{5n}{36}$$

Part b) To calculate the $Cov(Xi, Yj)$, we only need to consider the case where $i = j$, a specific roll of the dice, because when $i \neq j$, we are considering the X_i and Y_j from two different rolls of a dice, which are actually independent from each other, and therefore their covariance would be 0.
For $i = j$, the joint PDF of X_i and Y_j, $p(x_i, y_j)$ would be: p(0, 0) = 4/6 (the probability that neither 1 nor 2 shows in a roll of dice), p(0,1) = 1/6, p(1,0) = 1/6, p(1,1) = 0 (the probability that both 1 and 2 show in a roll of dice, which is not possible). So $E[X_iY_j]=0$, because X_iY_j is always equal to zero, and $E[X_i]E[Y_j]$ = 1/36. So we have the $Cov(Xi, Yj)$ = 0 – 1/36 = – 1/36.

See the example in Slide 11 of Lecture 18 on Covariance of 2 continuous random variables.

Hazard Rate: We discussed the importance of choosing intervals when solving questions on hazard rates and failure density functions based on the real data. Follow carefully, how the intervals are derived in the following two examples. In the first example, the intervals are fixed and both the intervals and number of failures are directly given by the data. But in the second example, the intervals are not equal and both the intervals and number of failures per interval are derived from the operating hours:

Example 1:

Table 4.3 Failure data for 172 hypothetical components

Time interval, hr	*Failures in the interval*
0–1,000	59
1,001–2,000	24
2,001–3,000	29
3,001–4,000	30
4,001–5,000	17
5,001–6,000	13
	Total 172

Table 4.4 Failure rates of hypothetical component

Time interval, hr	*Failure density* $f_d(t)(\times 10^{-4})$	*Hazard rate* $z_d(t)(\times 10^{-4})$
0–1,000	$\frac{59}{172 \times 10^3} = 3.43$	$\frac{59}{172 \times 10^3} = 3.43$
1,001–2,000	$\frac{24}{172 \times 10^3} = 1.40$	$\frac{24}{113 \times 10^3} = 2.12$
2,001–3,000	$\frac{29}{172 \times 10^3} = 1.69$	$\frac{29}{89 \times 10^3} = 3.26$
3,001–4,000	$\frac{30}{172 \times 10^3} = 1.74$	$\frac{30}{60 \times 10^3} = 5.00$
4,001–5,000	$\frac{17}{172 \times 10^3} = 0.99$	$\frac{17}{30 \times 10^3} = 5.69$
5,001–6,000	$\frac{13}{172 \times 10^3} = 0.76$	$\frac{13}{13 \times 10^3} = 10.00$

Example 2:

A 10,000-hr life test on a sample group of 15 electric motors produced the following data:

Motor Number	Hours of Operation
1-6	10,000
7-10	8,000
11	10,000
12-14	6,000
15	2,000

Assuming that all the motors have failed by the end of the test, construct a table showing for each time interval, the failure density per hour $f_d(t)$ $(\times 10^{-5})$ and the hazard rate per hour $z_d(t)$ $(\times 10^{-5})$ for the data. Plot the $f_d(t)$ and $z_d(t)$ in one graph. **Hint:** See Lecture 20 examples.

Solution: We first identify the number of failures occurred in each interval from the information provided on the hours of operation as follows. For example, for motor numbers 7-10, the hours of operation or lifetime is 8,000, so they all will fail by the end of the time interval 6000 - 8000.

Time Interval	Number of Failures	Motor Numbers
0 – 2000	1	15
2000 – 6000	3	12, 13, 14
6000 – 8000	4	7, 8, 9, 10
8000 – 10000	7	1, 2, 3, 4, 5, 6, 11

We can then derive the failure density and hazard rate per hour for each interval using the approach shown in Lecture 20, as follows. Please note that the total number of motors is 15, and the time intervals are not equal in this example.

Time Interval	Number of Failures	Failure Density $f_d(t)$ $(\times 10^{-5})$	Hazard Rate $h_d(t)$ or $z_d(t)$ $(\times 10^{-5})$
0 – 2000	1	$\frac{1}{15\times2000} = 3.33$	$\frac{1}{15\times2000} = 3.33$
2000 – 6000	3	$\frac{3}{15\times4000} = 5$	$\frac{3}{14\times4000} = 5.35$
6000 – 8000	4	$\frac{4}{15\times2000} = 13.3$	$\frac{4}{11\times2000} = 18.2$
8000 – 10000	7	$\frac{7}{15\times2000} = 23.3$	$\frac{7}{7\times2000} = 50$

Hypothesis Testing with Continuous Random Variables:

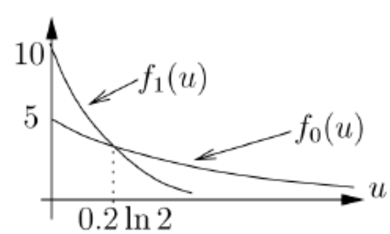
Example 1:

2. If hypothesis H_0 is true, the pdf of X is exponential with parameter 5 while if hypothesis H_1 is true, the pdf of X is exponential with parameter 10.
(a) Sketch the two pdfs.
(b) State the maximum-likelihood decision rule in terms of a threshold test on the observed value u of the random variable X instead of a test that involves comparing the likelihood ratio $\Lambda(u) = f_1(u)/f_o(u)$ to 1.
(c) Determine the probabilities of false-alarm and missed detection for the maximum-likelihood decision rule of part(b).
(d) The Bayesian (minimum probability of error) decision rule compares $\Lambda(u)$ to π_o/π_1. Show that this decision rule also can be stated in terms of a threshold test on the observed value u of the random variable X.
(e) If $\pi_o = 1/3$, determine is the average probability of error of the Bayesian decision rule.

(f) What is the average error probability of a decision rule that always decides H_1 is the true hypothesis, regardless of the value taken on by X?
(g) Show that if $\pi_o > 2/3$, the Bayesian decision rule always decides that H_0 is the true hypothesis regardless of the value taken on by X. Determine the average probability of error for the maximum-likelihood rule when $\pi_o > 2/3$.

Solution:

2. (a)



(b) $\Lambda(u) = \dfrac{f_1(u)}{f_o(u)} = \dfrac{10e^{-10u}}{5e^{-5u}} = 2e^{-5u}$ Note that $\Lambda(u) > 1$ for $u < 0.2 \ln 2$. Thus, the likelihood ration test is equivalent to deciding in favor of H_1 if the observed value of X is smaller than the threshold 0.2 ln 2.

(c) $P_{FA} = \int_{\Gamma_1} f_o(u)\,du = \int_0^{0.2\ln 2} 5e^{-5u}\,du = \dfrac{1}{2}$ $\qquad P_{MD} = \int_{\Gamma_o} f_1(u)\,du = \int_{0.2\ln 2}^{\infty} 10e^{-10u}\,du = \dfrac{1}{4}$

(d) $\Lambda(u) = 2e^{-5u} > \frac{\pi_o}{\pi_1}$ for $u < 0.2\ln\left(\frac{2\pi_1}{\pi_o}\right) = 0.2\ln 2 + 0.2\ln\left(\frac{\pi_1}{\pi_o}\right) = \xi$. Thus, the minimum-error-probability decision rule is equivalent to deciding in favor of H_1 if the observed value of X is smaller than ξ. That is, $\xi < 0$ if $\pi_o > 2\pi_1$ … if $\pi_o > 2/3$.

(e) If $\pi_o = 1/3$, then $\xi < 0.2\ln 4$.

$$P_{FA} = \int_0^{0.2\ln 4} 5e^{-5u}du = \frac{3}{4}$$

$$P_{MD} = \int_{0.2\ln 4}^{\infty} 10e^{-10u}du = \frac{1}{16}$$

The average error probability is $P(E) = \frac{1}{3}P_{FA} + \frac{2}{3}P_{MD} = \frac{7}{24}$. Note that because $\pi_o < \pi_1$, the Bayesian decision rule allows P_{FA} to increase in return for a decrease in P_{MD} because the latter is weighted more heavily.

(f) If the decision rule always decides H_1 is the true hypothesis it makes errors if and only if H_0 is the true hypothesis. Hence, $P(E) = \pi_o$.

(g) When $\pi_o > 2/3$, the threshold ξ is less than 0. Because X takes on nonnegative values, it is always larger than the threshold, and hence the decision is always H_0. The average error probability is π_1, and because this is the minimum-error-probability rule, we cannot do any better than this. Note that $\pi_1 < 1/3$.

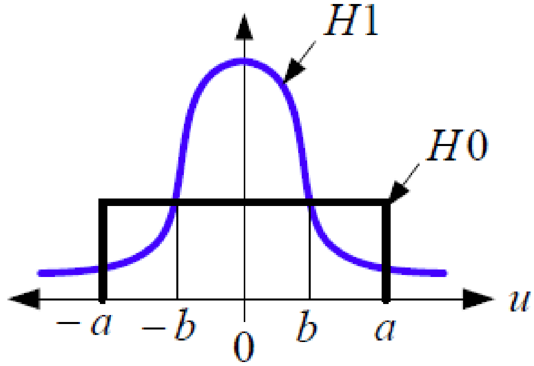
When $\pi_o > 2/3$, it follows that $\pi_o > 2\pi_1$. The average probability of error for the maximum-likelihood rule is $\pi_o(1/2) + \pi_1(1/4) > 2\pi_1(1/2) + \pi_1(1/3) = 1.25\pi_1$.

Example 2:

An observation X is drawn from a standard normal distribution (i.e. $N(0,1)$) if hypothesis H_1 is true and from a uniform distribution with support $[-a,a]$ if hypothesis H_0 is true. As shown in the figure below (under part (b)), the pdfs of the two distributions are equal when $|u| = b$.

(a) Express the maximum likelihood (ML) decision rule in a simple way, in terms of the observation X and the constants a and b.

(b) Shade and label the regions in the figure below such that the area of one of the regions is $p_{false\ alarm}$ and the area of the other region is p_{miss}.



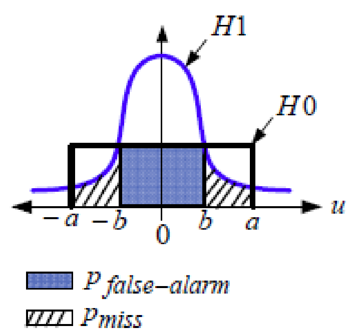
(c) Express $p_{false\ alarm}$ and p_{miss} for the ML decision rule in terms of the constants a, b, and the Φ function or Q function with positive arguments.

(d) **[6 points]** Determine the maximum *á posteriori* probability (MAP) rule when $a = \frac{3}{2}$, $b = 0.6$, and the probability of hypothesis $H1$ being true is. $\pi_1 = \frac{\sqrt{2\pi}}{3+\sqrt{2\pi}}$.

Solution:

(a) From the figure, the pdf for H_1 is smaller than the pdf for H_0 precisely when $b < |u| < a$. Thus, the ML rule is given by:

$$\widehat{H} = \begin{cases} H_0 & b < |X| < a \\ H_1 & \text{otherwise} \end{cases}$$



(b)

(c) These probabilities are calculated as follows:

$$\begin{aligned} p_{false\ alarm} &= \frac{2b}{2a} = \frac{b}{a} \\ p_{miss} &= 2(\Phi(a) - \Phi(b)) = 2(Q(b) - Q(a)) \end{aligned}$$

(d) Given π_1, one obtains

$$\begin{aligned} \pi_0 &= 1 - \pi_1 = \frac{\sqrt{3}}{3 + \sqrt{2\pi}} \\ \frac{\pi_0}{\pi_1} &= \frac{3}{\sqrt{2\pi}} \end{aligned}$$

The LRT gives us

$$\begin{aligned} \frac{3}{\sqrt{2\pi}} e^{\frac{-u^2}{2}} &= \frac{3}{\sqrt{2\pi}} \\ e^{\frac{-u^2}{2}} = 1 \end{aligned}$$

Thus, the MAP rule is given by

$$\hat{H} = \begin{cases} H1 & u > 0 \\ H0 & otherwise \end{cases}$$