Problem 1

- Consider a nonlinear amplifier whose input X and the output Y are related by its transfer characteristic:

$$Y = \begin{cases} X^{1/2}, & X \geq 0, \\ -|X|^{1/2}, & X < 0. \end{cases}$$

- Assuming that $X \sim N(0,1)$ compute the pdf of Y.

Problem 1 - Solution

- X is a standard normal distribution.
- To find the pdf of a function of a random variable, we first find the CDF (here $F_Y(y) = P(Y \le y)$), then differentiate it to get the pdf.

- For $y \ge 0$, $P(Y \le y) = P(\sqrt{X} \le y) = P(X \le y^2)$

$$= F_z(y^2)$$

where $F_Z(x)$ is the standard normal CDF.

- For $y < 0$, $P(Y \le y) = P(-\sqrt{|X|} \le y) = P(|X| \le y^2)$

$$= P(X \ge -y^2)$$

$$= 1 - F_z(-y^2) = F_z(y^2).$$

Problem 1 – Solution, Cont'd

- By differentiating $F_Y(y) = F_z(y^2)$, we get the pdf of Y:

$$f_Y(y) = 2|y| f_Z(y^2) = \frac{2|y|}{\sqrt{2\pi}} e^{-y^4/2}$$

- Remember that $f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$ is the pdf of the standard normal distribution $N(0,1)$.

Problem 2

- The joint density of X and Y is given by:

$$f(x,y) = C(y-x)e^{-y} \qquad -y < x < y,\ 0 < y < \infty$$

a) Find C.
b) Find the density function of X
c) Find the density function of X
d) Find $E[X]$.
e) Find $E[Y]$.
f) Find $P(X+Y<3)$, **for $X > 0$.**

Problem 2 - Solution

- In this solution, we will make use of the identity

$$\int_0^\infty e^{-x} x^n \; dx = n!$$

Which follows because $e^{-x} x^n / n!, x > 0$, is the density function of a gamma random variable with parameters $n+1$ and λ and must thus integrate to 1.

a) $$1 = C\int_0^\infty e^{-y} \int_{-y}^{y} (y - x) dx \, dy$$

$$= C\int_0^\infty e^{-y} 2y^2 dy = 4C$$

hence C=1/4

Problem 2 – Solution, Cont'd

b) Since the joint density is nonzero only when $y>x$ and $y>-x$, we have, for $x>0$,

$$f_X(x) = \frac{1}{4}\int_x^\infty (y-x)e^{-y}\,dy$$

$$= \frac{1}{4}\int_0^\infty ue^{-(x+u)}\,du$$

$$= \frac{1}{4}e^{-x}$$

for $x<0$

$$f_X(x) = \frac{1}{4}\int_{-x}^\infty (y-x)e^{-y}\,dy$$

$$= \frac{1}{4}\Big[-ye^{-y} - e^{-y} + xe^{-y}\Big]_{-x}^{\infty}$$

$$= (-2xe^{x} + e^{x})/4$$

Problem 2 – Solution, Cont'd

c) $f_Y(y) = \frac{1}{4}e^{-y}\int_{-y}^{y}(y-x)dx = \frac{1}{2}y^2e^{-y}$

d) $E[X] = \frac{1}{4}\left[\int_0^\infty xe^{-x}\,dx + \int_{-\infty}^0(-2x^2e^x + xe^x)dx\right]$

$$= \frac{1}{4}\left[1 - \int_0^\infty(2x^2e^{-x} + xe^{-x})dx\right]$$

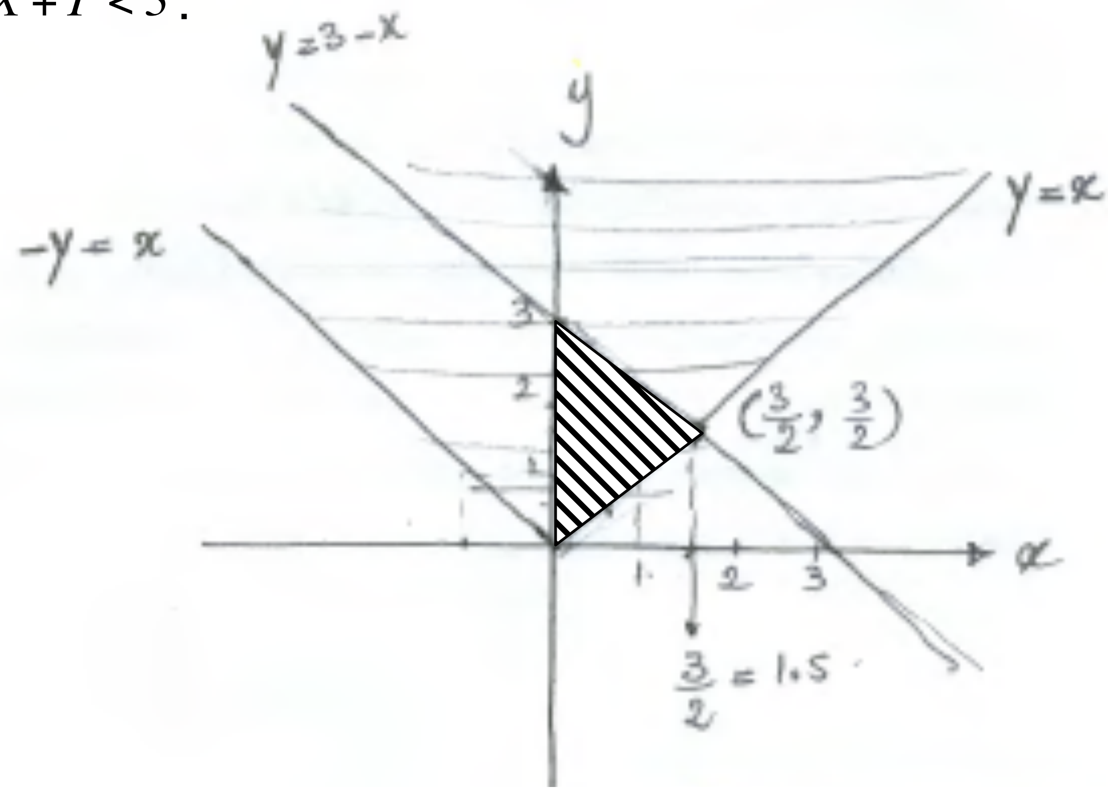
$$= \frac{1}{4}[1 - 4 - 1] = -1$$

e) $E[Y] = \frac{1}{2}\int_0^\infty y^3e^{-y}dy = 3$

Problem 2 – Solution, Cont'd

f. To find $P(X+Y<3)$, we first draw the line *X + Y = 3.*

For X > 0 :The region of integral is the triangle highlighted below, which is the intersection of regions $0<x<y, 0<y<\infty$ and $X+Y<3$:



Problem 2 – Solution, Cont'd

f. $P(X+Y<3)=\int_0^{1.5}\int_x^{3-x} f_{XY}(x,y)\,dy\,dx$

$$=\int_0^{1.5}\int_x^{3-x}\frac{1}{4}e^{-y}(y-x)\,dy\,dx$$

$$=\frac{1}{4}\int_0^{1.5}\left[\int_x^{3-x} ye^{-y}-xe^{-y}\,dy\right]dx$$

$$=\frac{1}{4}\int_0^{1.5} e^{-x}+2\mathrm{e}^{(x-3)}(x-2)\,dx=0.10155$$

Problem 2 – Solution, Cont'd

f. For X < 0 :The region of integral would be the intersection of regions $-y < x < 0, 0 < y < \infty$ and $X + Y < 3$: