ECE 313 - Section B
Final Exam
Fall 2013

Name: ______________________________

NetID: ______________________________

- **Be sure that your exam booklet has 12 pages.**
- **Write your name at the top of each page.**
- This is a **closed book** exam.
- You may consult your three 8.5" x 11" sheets of notes.
- No calculators, cell phones, PDAs, tablets, or laptop computers are allowed.
- **Please show all your work.** Answers without appropriate justification will receive **very little** or **no credit**.
- If you need extra space, use the back of the previous page.

Problem 1	________________	(40 pts)
Problem 2	________________	(15 pts)
Problem 3	________________	(20 pts)
Problem 4	________________	(15 pts)
Problem 5	________________	(10 pts)
Problem 6	________________	(10 pts)
TOTAL	________________	(110 pts)

Problem 1 (40 pts) – For each of the following parts, provide a short answer in the space provided, or choose the statement that is TRUE. Show your work and a justification for your answer to get partial credit.

Part A (3 pts): A component has a constant hazard/failure rate. Which distribution best models the time to failure of the component? Write the failure density function $f(t)$ and reliability function $R(t)$ for it.

Part B (4 pts) We have three components in series, each with a constant failure rate of λ, write the reliability function $R(t)$, the hazard rate $h(t)$, and the failure density function $f(t)$ of the system.

Part C (4 pts): The CPU time requirement of a typical program measured in minutes is found to follow a three stage Erlang distribution with $\lambda = \frac{1}{2}$. What is the probability that the CPU demand of a program will exceed 1 min?

Problem 1, continued:

Part D (4 pts): Consider a binary hypothesis testing problem where the prior probability of hypothesis H_0 is π_0 and the prior probability of hypothesis H_1 is π_1. Denote the probabilities of false alarm and missed detection for the ML decision rule by P_{FA}^{ML} and P_{MD}^{ML}, respectively. Similarly, denote the probabilities of false alarm and missed detection for the MAP decision rule by P_{FA}^{MAP} and P_{MD}^{MAP}, respectively. State whether the following statements are TRUE or FALSE. Include a short explanation:

(a) $P_{FA}^{ML} \cdot \pi_0 + P_{MD}^{ML} \cdot \pi_1 \geq P_{FA}^{MAP} \cdot \pi_0 + P_{MD}^{MAP} \cdot \pi_1$.

(b) If $\pi_0 = 0.5$ then $P_{MD}^{ML} = P_{MD}^{MAP}$.

Part E (5 pts): Let X and Y be two continuous random variables with joint density function:

$$f(x,y) = \begin{cases} 4e^{-2(x+y)}, & 0 \leq x < \infty \ , 0 \leq y < \infty \\ 0, & otherwise \end{cases}$$

(a) Are X and Y independent? Why?

(b) Are X and Y positively correlated, uncorrelated, or negatively correlated?

Problem 1, continued:

Part F (10 pts): Suppose n fair dice are independently rolled. Let:

$$X_k = \begin{cases} 1 & , if\ 1\ or\ 2\ show\ on\ the\ k^{th}\ die \\ 0 & , else \end{cases}$$
$$Y_k = \begin{cases} 1 & , if\ 3\ shows\ on\ the\ k^{th}\ die \\ 0 & , else \end{cases}$$

Let $X = \sum_{k=1}^{n} X_k$ and $Y = \sum_{k=1}^{n} Y_k$. State whether the following statements are TRUE or FALSE. Show your work to get credit for your answer.

(a) $Var(X_k) = \frac{2}{9}$

(b) $Var(X) = \frac{2n}{9}$

(c) $\mathrm{Cov}(\mathrm{X}_1, \mathrm{Y}_2) = -2$

(d) $\mathrm{Cov}(\mathrm{X}_1, \mathrm{Y}_1) \neq \mathrm{Cov}(\mathrm{X}_2, \mathrm{Y}_2)$

(e) $\rho_{X,Y} = \frac{18Cov(X,Y)}{\sqrt{10}n}$

Problem 1, continued:

Part G (10 pts): Suppose the number of job requests arriving at the Blue Waters supercomputing center is Poisson distributed with an average rate of λ requests per minute. Each job request arriving at the center is a GPU task with the probability of *p,* and demands more than 10 cores with the probability of *q* (These two events are independent from each other). We define the following random variables:

- *X*: the inter-arrival time of two consecutive jobs arriving at the center.
- *Y*: the random variable indicating whether a randomly picked job is a GPU task (Y = 1) or not (Y = 0).
- *Z*: the random variable indicating whether a randomly picked job needs more than 10 cores (Z = 1) or not (Z = 0).

Assume that *X*, *Y,* and *Z* are independent. Answer the following questions:

(a) What is the distribution of *X, Y,* and *Z*? Write their probability density function (pdf) or probability mass function (pmf) based on the parameters λ, *p*, and *q.*

X: ____________________________

Y: ____________________________

Z: ____________________________

(b) Suppose that $E[Y^2] = \frac{1}{3}$ and that $Var(3Y + 2Z) = 3$. Find p and q.

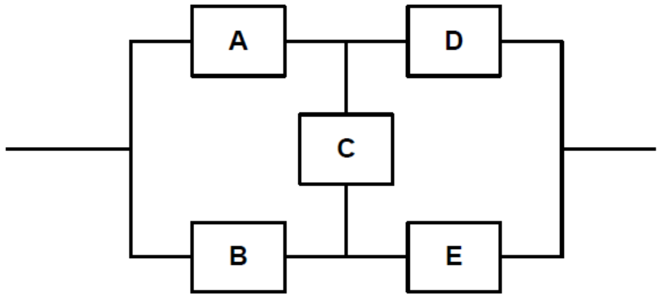
$p =$ _______________

$q =$ _______________

(c) Suppose that $\mathrm{E}\left[5X + 2Y + Z + \frac{1}{3}\right] = 2$. Find λ.

$\lambda =$ _______________

Problem 2 (15 pts) – Consider the non-series-parallel system of five independent components shown in the following figure:



Use the conditional probabilities to determine an expression for the system reliability as a function of component reliabilities. Assume that the reliability of each component is equal to R. **Hint:** Try to find a node around which you can simplify the analysis.

Problem 3 (20 pts): Suppose a certain firm has two plants A and B that produce chipsets and package them into boxes of 4. Assume that:

- 40% of the total boxes come from plant A, and 60% from plant B.
- The probability of a chipset produced by plant A being defective is 0.75, independent of all other events, and the probability of a chipset produced by plant B being defective is 0.5, independent of all other events.

A quality control crew randomly picks a box and tests each of the 4 chipsets independently for defects. Let X be the number of defective chipsets in the selected box. $\Lambda_{\mathrm{ML}}(X)$ is the ML decision rule to guess, based on the observation of X, whether the selected box came from plant A (H_0) or plant B (H_1).

Part A (2 pts): What are the prior probabilities of π_0 and π_1?

π_0 ____________

π_1 ____________

Part B (5 pts): What distribution best models the random variable X under each of the hypotheses H_0 and H_1? Write the probability mass function (pmf) of X under each of H_0 and H_1:

$Distribution\ under\ H_0 :\ P(X = k|H_0) =$ ______________________________

$Distribution\ under\ H_1 :\ P(X = k|H_1) =$ ______________________________

Part C (7 pts): Use the table provided below or the likelihood ratio test (LRT) to describe the $\Lambda_{\mathrm{ML}}(X)$ decision rule and find the threshold for X.

ML	X = 0	X = 1	X = 2	X = 3	X = 4
H_0					
H_1					

$\Lambda_{ML}(X) =$

$$\Lambda_{ML}(X) = \begin{cases} X \geq \_\_\_\_\_ & H_0 \\ X < \_\_\_\_\_ & H_1 \end{cases}$$

Part D (6 pts)**:** Calculate the probabilities of false alarm, miss detection, and error.

P_{FA}^{ML} = ______________________________

P_{MD}^{ML} = ______________________________ P_{ERROR}^{ML} = ______________________________

Problem 4 (15 pts) – Life insurance companies base their computations on mortality data like those presented in Table I. In each column of the table, *x* denotes the age in years and *lx* is the number of living people at age *x*.

Table I - American experience mortality table

x	*lx*	*x*	*lx*	*x*	*lx*
30	86,000	40	78,000	70	38,000
31	85,000	45	74,000	75	26,000
32	84,000	50	69,000	80	14,000
33	83,000	55	64,000	85	5,000
34	82,000	60	57,000	90	800
35	81,000	65	49,000	95	3

x = age in years; *lx* = number living at age *x*.

Use **time intervals of 10 years** to calculate the failure density function $f_d(t)$ and hazard rate $z_d(t)$. Fill in the associated columns in the following table. You do not need to simplify the expressions for $f_d(t)$ and $z_d(t)$.

Time Interval	Num of Living at the beginning of interval	Failure Density $f_d(t)$	Hazard Rate $h_d(t)$ or $z_d(t)$

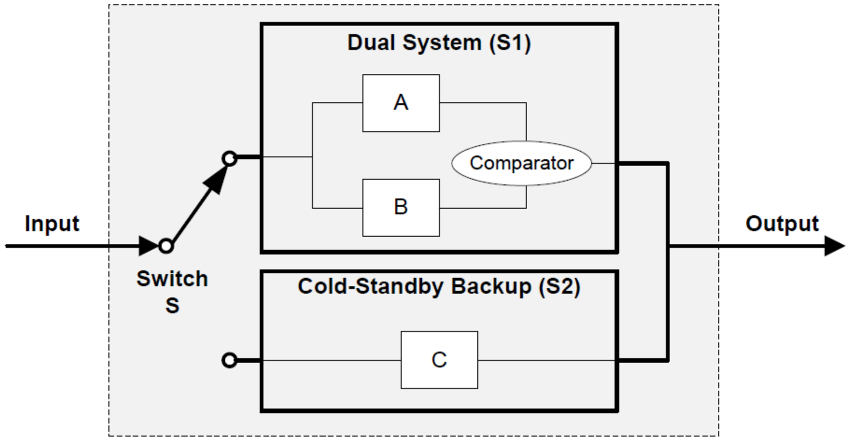
Problem 5 (10 pts) - The amount of time a typical ECE 313 student spends on the final project has a population mean $\mu = 3.10$ hours and standard deviation $\sigma = 0.40$ hours. If a random sample of 16 students is selected, use the central limit theorem to answer the following questions:

Part A (4 pts): Let $\bar{X}$ be the average time spent by the selected students. What is the mean and standard deviation of $\bar{X}$.

Part B (2 pts): What is the probability that the average time the selected students spend on the project will be at least 3 hours?

Part C (4 pts): There is an 85% chance that the average time the selected students spend on the project will be below N hours. What is N?

Problem 6 (10 pts) – Consider the following system composed of two subsystems S1 and S2. S1 is a dual system composed of two components A and B, where the failure of any of them will cause the failure of the subsystem S1. S2 is composed of a component C which acts as a backup (cold standby) of the subsystem S1 and will be powered on only after dual subsystem fails.



Assume that A and B are identical components and the switching circuit S and the comparator are perfect. We model the lifetime of the components A, B, and C with three independent random variables X_1, X_2, and X_3. Assume X_1 and X_2 are exponentially distributed with parameter λ and X_3 is exponentially distributed with parameter 3λ.

Part A (4 pts) – Find the reliability function and failure rate ($\lambda 1$) of the dual subsystem (S1).

Part B (6 pts) – What distribution best models the time to failure of the whole system? Use the results of part A to derive the reliability function of the system in terms of λ.

2-stage Hypoexponential Distribution:

$$f(t) = \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1}(e^{-\lambda_1 t} - e^{-\lambda_2 t}), \quad t > 0$$

$$F(t) = 1 - \frac{\lambda_2}{\lambda_2 - \lambda_1} e^{-\lambda_1 t} + \frac{\lambda_1}{\lambda_2 - \lambda_1} e^{-\lambda_2 t}, \quad t \geq 0$$

$$h(t) = \frac{\lambda_1 \lambda_2 (e^{-\lambda_1 t} - e^{-\lambda_2 t})}{\lambda_2 e^{-\lambda_1 t} - \lambda_1 e^{-\lambda_2 t}}, \quad t > 0$$

r-stage Erlang Distribution:

$$f(t) = \frac{\lambda^r t^{r-1} e^{-\lambda t}}{(r-1)!}, \quad t > 0, \lambda > 0, r = 1,2,...$$

$$F(t) = 1 - \sum_{k=0}^{r-1} \frac{(\lambda t)^k}{k!} e^{-\lambda t}, \quad t \geq 0, \lambda > 0, r = 1,2,...$$

$$h(t) = \frac{\lambda^r t^{r-1}}{(r-1)! \sum_{k=0}^{r-1} \frac{(\lambda t)^k}{k!}}, \quad t > 0, \lambda > 0, r = 1,2,...$$

Table 2.3 Area $\Phi(x)$ under the Standard Normal Curve to the Left of x

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5597	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8557	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998