ECE 313: Problem Set 11: Problems and Solutions
Working with joint densities

Due: Wednesday November 28 at 4 p.m.
Reading: *313 Course Notes* Sections 4.5-4.7

1. **[Sum of two independent geometric random variables]**
Let X be a geometric random variable with parameter p, and Y be a geometric random variable with parameter q, with X and Y being independent. Let $S = X + Y$.

   (a) Suppose $0 < q < p < 1$. Find the pmf of S using convolution. Simplify your expression as much as possible (in particular, don't leave the answer as a summation.)
   Solution: Note that $p_X(k) = p(1-p)^{k-1}$ and $p_Y(k) = q(1-q)^{k-1}$, for $k \geq 1$. Therefore, for $k \geq 2$

$$p_S(k) = \sum_{j=1}^{k-1} p_X(j)p_Y(k-j) = \sum_{j=1}^{k-1} p(1-p)^{j-1}\, q(1-q)^{k-j-1}$$
$$= pq(1-q)^{k-2}\sum_{m=0}^{k-2}(1-p)^m\,(1-q)^{-m} = pq(1-q)^{k-2}\,\frac{1-\left(\frac{1-p}{1-q}\right)^{k-1}}{1-\frac{1-p}{1-q}}$$
$$= pq\,\frac{(1-q)^{k-1}-(1-p)^{k-1}}{p-q}$$

   (b) Now consider the special case of $p = q$. Find the pmf of S using convolution.
   Solution: In this case, for $k \geq 2$,

$$p_S(k) = \sum_{j=1}^{k-1} p_X(j)p_Y(k-j) = \sum_{j=1}^{k-1} p(1-p)^{j-1}\, p(1-p)^{k-j-1} = p^2(1-p)^{k-2}\sum_{j=1}^{k-1} 1 = (k-1)p^2(1-p)^{k-2}$$

   (c) Verify your answer to part (b) using the negative binomial distribution.
   Solution: It is easy to see that S of part (b) is equal to the number of trials it takes to see two ones in a sequence of independent Bernoulli (p) trials, and therefore it has the negative binomial distribution with $r = 2$.

2. **[Sum of two independent continuous-type random variables]**
Suppose X and Y have the joint pdf

$$f_{X,Y}(u,v) = \begin{cases} 2v & \text{if } 0 \leq u \leq 1,\ 0 \leq v \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Let $S = X + Y$. Find the pdf of S, i.e., find $f_S(s)$.

Hint: You may want to consider the cases $s < 0$, $0 \leq s < 1$, $1 \leq s \leq 2$, and $s > 2$ separately.

Solution: Clearly $f_S(s) = 0$ for $s < 0$ and $s > 2$. For $0 \leq s \leq 2$,

$$f_S(s) = \int_{u=-\infty}^{\infty} f_{X,Y}(u, s-u)du$$

Now $f_{X,Y}(u, s-u) = 2(s-u)$ if $0 \leq u \leq 1$ and $0 \leq s-u \leq 1$ (i.e., $s-1 \leq u \leq s$), and is equal to 0 otherwise. Therefore, for $0 \leq s < 1$,

$$f_S(s) = \int_{u=0}^{s} 2(s-u)du = \int_{t=0}^{s} 2tdt = s^2$$

For $1 \leq s \leq 2$,

$$f_S(s) = \int_{u=s-1}^{1} 2(s-u)du = \int_{t=s-1}^{1} 2tdt = 1-(s-1)^2$$

Putting it all together, we have:

$$f_S(s) = \begin{cases} s^2 & \text{if } 0 \leq s < 1 \\ 1-(s-1)^2 & \text{if } 1 \leq s \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

It is easy to check that f_S integrates to 1.

3. **[Sum of two dependent continuous-type random variables]**
We know from PS 10 that the random variables X and Y with the following joint pdf are dependent.

$$f_{X,Y}(u,v) = \begin{cases} 6e^{-(u+2v)} & \text{if } 0 \leq u < v \\ 0 & \text{otherwise} \end{cases}$$

Find the pdf of the sum $S = X + Y$,

Solution: Clearly $f_S(s) = 0$ for $s < 0$. For $0s \geq 0$,

$$f_S(s) = \int_{u=-\infty}^{\infty} f_{X,Y}(u, s-u)du$$

Now $f_{X,Y}(u, s-u) = 6e^{-(u+2(s-u))}$ if $u \geq 0$ and $u < s-u$ (i.e., $u \leq \frac{s}{2}$), and is equal to 0 otherwise. This means that for $s \geq 0$,

$$f_S(s) = \int_{u=0}^{\frac{s}{2}} 6e^{-2s+u}du = 6\ e^{-2s} \int_{u=0}^{\frac{s}{2}} e^u du = 6\ e^{-2s}\ (e^{\frac{s}{2}} - 1)$$

Therefore

$$f_S(s) = \begin{cases} 6(e^{-\frac{3s}{2}} - e^{-2s}) & \text{if } s \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

It is easy to check that f_S integrates to 1.

4. **[Rayleigh fading and selection diversity]**
The signal strength received at a base-station from a cellular phone in a dense urban environment can be modeled quite well by a Rayleigh random variable. When a cellular phone user is around the midpoint between two base-stations, some systems use *selection diversity*, where the base station with the larger received signal strength is used to decode the user's message. Let X and Y denote the signal strengths received at the two base-stations. Then

$$f_X(u) = f_Y(u) = \begin{cases} ue^{-\frac{u^2}{2}} & \text{if } u \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Assume that X and Y are independent.

(a) Find the mean signal strength at each base-station, i.e., find $E[X]$.
Hint: Convert the integral required to compute $E[X]$ into one that corresponds to finding the variance of a $N(0,1)$ random variable.
Solution: Using the hint

$$E[X] = \int_0^{\infty} x^2\ e^{-\frac{x^2}{2}}dx = \frac{1}{2}\int_{-\infty}^{\infty} x^2\ e^{-\frac{x^2}{2}}dx = \frac{\sqrt{2\pi}}{2}\int_{-\infty}^{\infty} x^2\ \frac{1}{\sqrt{2\pi}}\ e^{-\frac{x^2}{2}}dx = \frac{\sqrt{2\pi}}{2} = \sqrt{\frac{\pi}{2}} \approx 1.253$$

(b) Find the pdf of the signal strength chosen by the selection diversity system, i.e., find the pdf of $Z = \max(X, Y)$.

Solution: We first compute the CDF of Z as

$$F_Z(z) = P\{Z \le z\} = P\{\max(X,Y) \le z\} = P(X \le z, Y \le z) = F_X(z)\, F_Y(z) = \left(1 - e^{-\frac{z^2}{2}}\right)^2$$

for $z \ge 0$, and $F_Z(z) = 0$ for $z < 0$. Differentiating we get

$$f_Z(z) = 2\, \left(1 - e^{-\frac{z^2}{2}}\right)\, e^{-\frac{z^2}{2}}\, z = 2ze^{-\frac{z^2}{2}} - 2ze^{-z^2}$$

for $z \ge 0$, and $f_Z(z) = 0$ for $z < 0$. It is easy to check that f_Z integrates to 1.

(c) Find the mean signal strength after selection diversity, i.e., find $E[Z]$.

Solution: Using the same technique as in part (a)

$$\begin{aligned}E[Z] &= \int_0^\infty 2\, z^2\, e^{-\frac{z^2}{2}} dz - \int_0^\infty 2\, z^2\, e^{-z^2} dz = \int_{-\infty}^\infty z^2\, e^{-\frac{z^2}{2}} dz - \int_{-\infty}^\infty z^2\, e^{-z^2} dz \\ &= \sqrt{2\pi} \int_{-\infty}^\infty z^2\, \frac{1}{\sqrt{2\pi}}\, e^{-\frac{z^2}{2}} dz - \sqrt{\pi} \int_{-\infty}^\infty z^2\, \frac{1}{\sqrt{\pi}}\, e^{-z^2} dz = \sqrt{2\pi} - \frac{\sqrt{\pi}}{2} \approx 1.62\end{aligned}$$

We see that $E[Z] > E[X]$ as expected.

5. **[Joint pdf of functions of random variables]**

Let X and Y be random variables, and define two other random variables as $W = \frac{1}{\sqrt{2}}(X - Y)$ and $Z = \frac{1}{\sqrt{2}}(X + Y)$.

(a) If X and Y are *independent* $N(0,1)$ random variables, then show that W and Z are also *independent* $N(0,1)$ random variables.

Solution: Applying Proposition 4.7.1, we see that

$$f_{W,Z}(w,z) = \frac{1}{|\det(A)|}\, f_{X,Y}(A^{-1}[wz]^\top)$$

where

$$A = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \qquad \det(A) = 1 \qquad A^{-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

Thus

$$\begin{aligned}f_{W,Z}(w,z) &= f_{X,Y}\left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} w \\ z \end{bmatrix}\right) = f_{X,Y}\left(\frac{w+z}{\sqrt{2}}, \frac{z-w}{\sqrt{2}}\right) \\ &= \frac{1}{2\pi} \exp\left(-\frac{(w+z)^2 + (z-w)^2}{4}\right) = \frac{1}{2\pi} \exp\left(-\frac{w^2+z^2}{2}\right) \\ &= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{w^2}{2}\right)\, \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right)\end{aligned}$$

This means that W and Z are independent $N(0,1)$ random variables.

(b) Now consider the case where X and Y are still independent, but $X \sim N(0,1)$ whereas $Y \sim N(0,2)$. Find the joint distribution of W and Z. Are W and Z independent?

Solution: Following the initial steps in part (a)

$$\begin{aligned}f_{W,Z}(w,z) &= f_{X,Y}\left(\frac{w+z}{\sqrt{2}}, \frac{z-w}{\sqrt{2}}\right) = \frac{1}{2\pi\sqrt{2}} \exp\left(-\frac{(w+z)^2}{4} - \frac{(z-w)^2}{8}\right) \\ &= \frac{1}{2\pi\sqrt{2}} \exp\left(-\frac{3w^2 + 2wz + 3z^2}{8}\right)\end{aligned}$$

This joint pdf does not factor into a product pdfs for W and Z. Therefore, W and Z are *not* independent in this case.