University of Illinois Fall 2012

ECE 313: Final Exam

Monday, December 17, 2012, 8:00 - 11:00 a.m.
151 Everitt Lab (Sections X and C) , 165 Everitt Lab (Section D), & 269 Everitt Lab (Section E)

1. **[16 points]** Let $S = X_1 + X_2 + \cdots + X_n$, where $n \geq 3$ and $X_1, X_2, \ldots, X_n$ are mutually independent, and, for each i, X_i has the Bernoulli distribution with parameter p.
   (a) (8 points) Find $P(S = 2|X_1 = 0, X_2 = 1)$. (Express your answer in terms of n and p.)
   (b) (8 points) Find $P(X_1 = 0, X_2 = 1|S = 2)$. You may express your answer as a fraction involving binomial coefficients and other factors; you do not need to simplify the answer further.

2. **[16 points]** A job arriving in a certain cloud computing system must be routed to one of eight servers. Due to the loads, the servers all have different rates. Routing the job to the server with the highest rate would require sampling the rates of all eight servers. Instead, the rates of three randomly selected distinct servers are sampled (all choices being equally likely) and the job is routed to the *sampled* server with the highest service rate.
   (a) (8 points) Let A be the event the job is assigned to the server with the highest service rate (among all eight servers). Find $P(A)$.
   (b) (8 points) Let B be the event the job is assigned to one of the four slowest servers. Find $P(B)$.

3. **[24 points]** *Noisy Geiger Counter.* A radioactive source emits X particles in one second, where X is a Poisson(λ) random variable. The number of emitted particles is measured by a noisy counter that might add or drop one from the count. Denote the output of the counter by the random variable Y. The probability the counter adds a count is the same as the probability it drops a count, which is equal to ϵ, with $0 < \epsilon < 0.5$. Assume the counter is smart enough not to produce negative counts, so if $X = 0$ the counter will not drop a count, but it could add a count with probability ϵ.
   (a) (6 points) Find the conditional pmf of Y, given $X = 0$.
   (b) (6 points) Find the conditional pmf of Y, given $X = \ell$, for a fixed $\ell \geq 1$.
   (c) (6 points) Find $P\{Y = 1\}$, which is the unconditional probability that $Y = 1$.
   (d) (6 points) Find $P(X = 1|Y = 1)$. Simplify your answer as much as possible.

4. **[20 points]** Given that a random variable X has the pdf:

$$f_X(u) = \begin{cases} \frac{1}{2}(2-u), & 0 < u < 2 \\ \\ 0, & \text{elsewhere} \end{cases}$$

   (a) (8 points) Find the CDF, $F_X(c)$. Be sure to specify it for all values of c.
   (b) (6 points) Determine P$\{X > 1\}$
   (c) (6 points) Determine P$\{X > 1 \mid X \leq 2\}$

5. **[14 points]** Let X be a continuous-type random variable with pdf $f_X(u) = a\cos(u)$ for $0 \leq u \leq b$ and zero else, where a, b are positve constants. It is known that $F_X(\pi/4) = \frac{1}{\sqrt{2}}$. Find the constants a and b and find $F_X(c)$ for all c.

6. **[18 points]** Let X and Y be independent random variables, where each is exponentially distributed with parameter one. Let $S = \max\{X, Y\} - \min\{X, Y\}$, or equivalently, $S = |X - Y|$.

   (a) (4 points) Find the support of the pdf of S, $f_S(c)$.

   (b) (6 points) Let $c \geq 0$. Carefully sketch and label the region in the upper quadrant of the $u - v$ plane such that $\{S \leq c\}$ is equivalent to (X, Y) being in the region.

   (c) (8 points) Find the pdf $f_S(c)$ for all c. (This can be done either using part (b) or by a different method.)

7. **[15 points]** Suppose random variables X and Y have the joint pdf:

$$f_{X,Y}(u, v) = \begin{cases} 2e^{u-2v}, & u \leq 0, \ v \geq 0 \\ 0, & \text{elsewhere} \end{cases}$$

   Find the joint CDF, $F_{X,Y}(a, b)$ for all real values of a and b.

8. **[24 points]** Suppose X, Y, and Z are random variables, each with mean zero and variance 20, such that $\mathrm{Cov}(X, Y) = \mathrm{Cov}(X, Z) = 10$ and $\mathrm{Cov}(Y, Z) = 5$. Be sure to show your work, as usual, for all parts below.

   (a) (6 points) Find $\mathrm{Cov}(X + Y, X - Y)$.

   (b) (6 points) Find Cov(3X+Z,3X+Y).

   (c) (6 points) Find $E[(X + Y)^2]$.

   (d) (6 points) Find $\widehat{E}[Y + Z|X = 3]$.

9. **[24 points]** Suppose X and Y are jointly Gaussian with $\mu_X = \mu_Y = 0$, $\sigma_X^2 = 1$, $\sigma_Y^2 = 4$, and $\rho_{XY} = 0.25$.

   (a) (6 points) Let $W = X - \alpha Y$. Find the value of α that makes X and W independent.

   (b) (6 points) Let α be chosen as in part (a), so that X and W are independent. Find the unconstrained MMSE estimator of X based on W, and the resulting MSE.

   (c) (6 points) Let $Z = 2X + Y + 2$. Find the mean and variance of Z.

   (d) (6 points) Find the unconstrained MMSE estimator of Y based on Z.

10. **[24 points]** Suppose that random variables X and Y have the joint pdf:

$$f_{XY}(u, v) = \begin{cases} 4u^2, & 0 < v < u < 1 \\ 0, & \text{elsewhere} \end{cases}$$

   (a) (6 points) Find $E[XY]$.

   (b) (6 points) Find $f_Y(v)$. Be sure to specify it for all values of v.

   (c) (6 points) Find $f_{X|Y}(u|v)$. Be sure to specify where it is undefined, and where it is zero.

   (d) (6 points) Find $E[X^2|Y = v]$ for $0 < v < 1$.

11. **[30 points]** (3 points per answer)
In order to discourage guessing, 3 points will be deducted for each incorrect answer (no penalty or gain for blank answers). A net negative score will reduce your total exam score.

(a) The random variables X and Y are independent if their joint pdf is:

TRUE	FALSE	
☐	☐	$f_{X,Y}(u,v) = \begin{cases} k(u^2+v^2) & 0 \le u \le 2, \quad 1 \le v \le 4 \\ 0 & \text{else} \end{cases}$
☐	☐	$f_{X,Y}(u,v) = \begin{cases} \frac{1}{v} & 0 \le u \le v \le 1 \\ 0 & \text{else} \end{cases}$

(b) Consider a binary hypothesis testing problem where the prior probabilities of the hypotheses H_0 and H_1 are π_0 and π_1, respectively. Denote the probabilities of false alarm and miss for the ML decision rule by $p^{\text{ML}}_{\text{false-alarm}}$ and $p^{\text{ML}}_{\text{miss}}$, respectively. Similarly, denote the probabilities of false alarm and miss for the MAP decision rule by $p^{\text{MAP}}_{\text{false-alarm}}$ and $p^{\text{MAP}}_{\text{miss}}$ respectively

TRUE	FALSE	
☐	☐	If $\pi_1 = \pi_0$, then $p^{\text{ML}}_{\text{false-alarm}} = p^{\text{MAP}}_{\text{false-alarm}}$
☐	☐	If $\pi_1 = 2\pi_0$, then $p^{\text{ML}}_{\text{false-alarm}} = 2p^{\text{ML}}_{\text{miss}}$

(c) Let A, B, C be independent events with $0 < P(A), P(B), P(C) < 1$.

TRUE	FALSE	
☐	☐	$P(AB\|C) \ge P(A\|BC)P(B)$
☐	☐	$P(AB\|C) = P(AB\|C^c)$
☐	☐	$P(AB\|C) \le P(C\|AB)P(AB)$

(d) Suppose X and Y have mean zero, variance one, and correlation coefficient $\rho = 0.5$.

TRUE	FALSE	
☐	☐	$\widehat{E}[Y\|X] = \frac{X}{2}$
☐	☐	$E[(Y - g(X))^2] \ge 0.5$ for any choice of the function g
☐	☐	$E[(Y - a - bX)^2] \ge 0.5$ for any choice of the constants a and b.