University of Illinois Fall 2012

ECE 313: Conflict Final Exam

Tuesday, December 18, 2012
1:30 p.m. — 4:30 p.m.
160 English Building

1. (a) The set of possible values of S is $\{n, n+1, \ldots, 2n\}$. For k in this set, $S = k$ if exactly $k - n$ of the n X_i's are equal to two. Hence

$$p_S(k) = p_B(k-n) = \binom{n}{k-n} p^{k-n}(1-p)^{n-(k-n)} = \binom{n}{k-n} p^{k-n}(1-p)^{2n-k}$$

for $n \leq k \leq 2n$.
ALTERNATIVELY, we see that $Y_i = X_i - 1$ is a Bernoulli random variable with parameter p, and $S = n + B$, where $B = Y_1 + \cdots + Y_n$, so B has the binomial distribution with parameters n and p. Hence, for $n \leq k \leq 2n$, $p_S(k) = p_B(k-n) = \binom{n}{k-n} p^{k-n}(1-p)^{n-(k-n)}$.

   (b) $E[X_1] = p \cdot 2 + (1-p) \cdot 1 = 1 + p$. Hence, $E[S] = nE[X_1] = n(1+p)$.
ALTERNATIVELY, $E[S] = E[B+n] = E[B] + n = np + n = n(1+p)$.

   (c) $\mathrm{Var}(X_1) = E[X_1^2] - E[X_1]^2 = 4p + (1-p) - (1+p)^2 = p(1-p)$. So, since the X_i's are independent (hence uncorrelated), $\mathrm{Var}(S) = n\mathrm{Var}(X_1) = np(1-p)$.
ALTERNATIVELY, $Var(S) = Var(B+n) = Var(B) = np(1-p)$.

2. (a)

$$F_X(c) = \begin{cases} 0, & c < 0 \\ \int_0^c 5\, e^{-5u}\, du = -e^{-5u}\,\Big|_0^c = 1 - e^{-5c}, & c > 0 \end{cases}$$

   (b) $\mathrm{P}\{X > 1\} = 1 - \mathrm{P}\{X \leq 1\} = 1 - F_X(1) = e^{-5}$

   (c) $\mathrm{P}\{X > 1 \mid X \leq 2\} = \frac{P\{X>1, X\leq 2\}}{P\{X \leq 2\}} = \frac{F_X(2) - F_X(1)}{F_X(2)} = \frac{e^{-5} - e^{-10}}{1 - e^{-10}}$

3. (a) No. For example, $P\{T > 1\} = 1 - F_T(1) = e^{-1}$, whereas

$$P(T > 2 | T > 1) = \frac{P(T > 2, T > 1)}{P\{T > 1\}} = \frac{P\{T > 2\}}{P\{T > 1\}} = \frac{1 - F_T(2)}{1 - F_T(1)} = \frac{e^{-4}}{e^{-1}} = e^{-3}$$

   (b) For $x > 0$

$$F_X(x) = P\{X \leq x\} = P\{T \leq \sqrt{x}\} = 1 - e^{-x}$$

and $f_X(x) = e^{-x}$, i.e., x has an exponential distribution.

   (c) Yes, as we proved in class, the exponential distribution is memoryless. For any $s, t > 0$,

$$P(T > t+s | T > t) = \frac{P(T > t+s, T > t)}{P\{T > t\}} = \frac{P\{T > t+s\}}{P\{T > t\}} = \frac{e^{-(t+s)}}{e^{-t}} = e^{-s} = P\{T > s\}.$$

4. (a) The time between emails is an exponential random variable with mean $1/\lambda = 1/5$, so you expect to wait 2/5 hours.

   (b) The number of emails in a 50 minute period is a Poisson random variable with mean $\hat{\lambda} = \lambda(50/60) = 25/6$, so you expect 25/6 emails.

(c) The number of emails in a 50 minute period is a Poisson random variable with mean $\hat{\lambda} = \lambda(50/60) = 25/6$, so $P\{5 \text{ emails in 50 mintes}\} = \frac{e^{-\hat{\lambda}}\hat{\lambda}^5}{5!} = \frac{e^{-25/6}(25/6)^5}{5!}$.

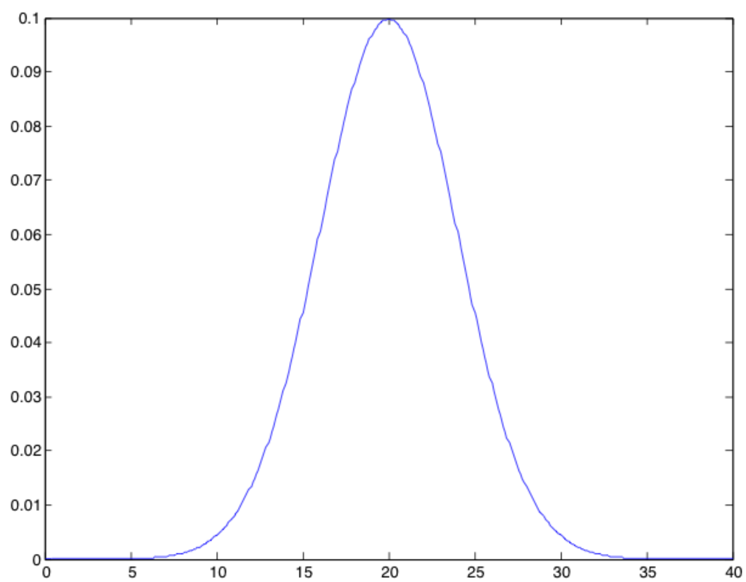
(d) Let R_1 be the arrival time of the first real news email, and let F_1 be the arrival time of the first fake email.

$$P\{\text{first email you receive is fake}\} = P\{F_1 < R_1\} = \int_0^\infty P\{F_1 < R_1 | R_1 = u\} f_{R_1}(u)du$$
$$= \int_0^\infty \left(1 - e^{-10u}\right) 5e^{-5u} du = 5\left[\frac{e^{-5u}}{-5} - \frac{e^{-15u}}{-15}\right]\Bigg|_0^\infty = 5\left(\frac{1}{5} - \frac{1}{15}\right) = \frac{10}{15} = \frac{2}{3}.$$

5. (a) The Gaussian density, $\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(u-\mu)^2}{2\sigma^2}\right)$ is maximized at $u = \mu$. The maximum value is $f_X(\mu) = \frac{1}{\sqrt{2\pi\sigma^2}} = \frac{1}{\sqrt{32\pi}} \approx 0.1$.

(b) $f_X(u) = (0.5)f_X(\mu)$ if the exponential term in the pdf at u is equal to 0.5. That is, if $\frac{(u-\mu)^2}{2\sigma^2} = \ln(2)$ or $(u-\mu)^2 = 2(\ln 2)\sigma^2$ or $u = \mu \pm \sigma\sqrt{2\ln 2} = 16 \pm 4\sqrt{2\ln 2} \approx 16 \pm 4.7$.

(c) A matlab plot is shown. Important features are that the pdf is centered at 20, the maximum is around 0.1, and the shape is a bell curve with halfwidth around 5.



(d) $P\{|X| \geq 30\} = P\{X \geq 30\} + P\{X \leq -30\} = P\{\frac{X-20}{4} \geq \frac{30-20}{4}\} + P\{\frac{X-20}{4} \leq \frac{-30-20}{4}\} = Q(2.5) + 1 - Q(-12.5) = Q(2.5) + Q(12.5) \approx Q(2.5) \approx 0.0062$.

6. By definition, $\widehat{\theta}_{ML}(10)$ is the value of θ that maximizes the likelihood of $X = 10$. The likelihood of $X = 10$ is $f_\theta(10) = \left(\frac{10}{\theta}\right) e^{-\frac{100}{2\theta}}$, and the log likelihood is $\ln(10) - \ln(\theta) - \frac{50}{\theta}$. Differentiation with respect to θ yields $\frac{d \ln f_\theta(10)}{d\theta} = -\frac{1}{\theta} + \frac{50}{\theta^2}$. This derivative is zero for $\theta = 50$, and it is positive for $\theta < 50$ and negative for $\theta > 50$. Hence, $\widehat{\theta}_{ML}(10) = 50$.
(Note: If θ is replaced by σ^2, then f is the Rayleigh pdf with parameter σ^2. The same reasoning as above shows that, in general, for observation $X = u$, $\widehat{\theta}_{ML} = \widehat{(\sigma^2)}_{ML} = \frac{u^2}{2}$.)

7. (a) The range of possible values of X is the interval $[0, 1]$. By inspection (or using the definition of conditional density), if $0 \leq u \leq 1$, we see that given $X = u$, the conditional distribution of Y is the uniform distribution over the interval $[0, u]$. Therefore, $E[Y^2 | X = u] = \int_0^u \frac{1}{u} v^2 dv = \frac{u^2}{3}$.

(b) By the same observation used in part (a), $E[Y|X=u]=\frac{u}{2}$. Since this is a linear function of u, it is also equal to $\widehat{E}[Y|X=u]$. That is, $\widehat{E}[Y|X=u]=\frac{u}{2}$. (This result can be worked out using the formula for $\widehat{E}[Y|X=u]$ as well.)

8. (a) First note that $\mathrm{Cov}(X,Y)=\sigma_X\sigma_Y\rho=\frac{1}{4}$. Therefore

$$\mathrm{Cov}(X,W)=\mathrm{Cov}(X,X+\alpha Y+\beta)=\mathrm{Var}(X)+\alpha\mathrm{Cov}(X,Y)=1+\frac{\alpha}{4},$$

and $\mathrm{Cov}(X,W)=0$ if $\alpha=-4$, no matter what value β takes. Since X and W are jointly Gaussian, $\alpha=-4$ (and β arbitrary) also makes them independent.

(b) $E[Z]=E[4X+2Y+2]=4E[X]+2E[Y]+3=7$, and

$$\mathrm{Var}(Z)=16\mathrm{Var}(X)+4\mathrm{Var}(Y)+16\mathrm{Cov}(X,Y)=16+16+4=36$$

(c) Since Y and Z are jointly Gaussian

$$E[Y|Z=11]=L^*(11)=E[Y]+\frac{\mathrm{Cov}(Y,Z)}{\mathrm{Var}(Z)}(11-E[Z])$$

Now

$$\mathrm{Cov}(Y,Z)=\mathrm{Cov}(Y,4X+2Y+2)=4\mathrm{Cov}(Y,X)+2\mathrm{Var}(Y)=1+8=9$$

and therefore

$$E[Y|Z=11]=L^*(11)=0+\frac{9}{36}(11-7)=1$$

(d) Given, $Z=11$, Y is Gaussian with mean $E[Y|Z=11]=1$ as calculated above, and variance

$$\sigma_e^2=\text{MSE of } L^*(11)=\mathrm{Var}(Y)-\frac{(\mathrm{Cov}(Y,Z))^2}{\mathrm{Var}(Z)}=4-\frac{9^2}{36}=4-\frac{9}{4}=\frac{7}{4}$$

Therefore

$$E[Y^2|Z=11]=1+\frac{7}{4}=\frac{11}{4}$$

9. (a)

$$f_Y(v)=\begin{cases}\int_{-v}^{\infty}\frac{1}{2}\,e^{-u}\,du=-\frac{1}{2}\,e^{-u}\,|_{-v}^{\infty}=\frac{1}{2}\,e^{v}, & v<0\\ \int_{v}^{\infty}\frac{1}{2}\,e^{-u}\,du=\frac{1}{2}\,e^{-v}, & v>0\end{cases}\quad=\quad\frac{1}{2}\,e^{-|v|}$$

(b) $E\left[\frac{1}{X}\right]=\int_0^\infty\int_{-u}^{u}\frac{1}{u}\frac{1}{2}e^{-u}dvdu=\int_0^\infty e^{-u}du=1$

10. (a)

$$f_X(u)=\begin{cases}\int_u^1 2\,dv=2(1-u), & 0<u<1\\ 0, & \text{elsewhere}\end{cases}$$

$$\Longrightarrow f_{Y|X}(v|X=u)=\frac{1}{1-u},\qquad 0<u<1,\ u\le v<1$$

That is, given $X=u$, Y is uniformly distributed over the interval $[u,1]$, as can also be seen by inspection. Therefore,

$$\textit{and}\quad g^*(u)=E[Y|X=u]=\int_u^1\frac{v}{1-u}\,dv=\frac{1+u}{2},\qquad 0<u<1$$

(b)

$$E[Y^2|X=u] = \int_u^1 \frac{v^2}{1-u}\,dv = \frac{1+u+u^2}{3}$$

$$\implies MMSE = Var(Y|X=u) = E[Y^2|X=u] - (E[Y|X=u])^2 = \frac{(1-u)^2}{12}$$

(c) Since the unconstrained estimator is linear in u, the best linear estimator is:

$$E[Y|X=u] = \frac{1+u}{2}$$

11. (a) False, False, True
    (b) True, False, False
    (c) True, True
    (d) True, False