University of Illinois Fall 2012

ECE 313: Conflict Final Exam

Tuesday, December 18, 2012
1:30 p.m. — 4:30 p.m.
160 English Building

1. **[20 points]** Consider n random variables $X_1, X_2, \ldots, X_n$, which are independent and identically distributed, such that $X_1 = 1$ with probability $1-p$ and $X_1 = 2$ with probability p. Let $S = X_1 + X_2 + \cdots + X_n$.

   (a) (8 points) Find the pmf of S.

   (b) (6 points) Find the mean of S.

   (c) (6 points) Find the variance of S.

2. **[20 points]** Given that a random variable X has the pdf:

$$f_X(u) = \begin{cases} 5\ e^{-5u}, & u > 0 \\ 0, & \text{elsewhere} \end{cases}$$

   (a) (8 points) Find the CDF, $F_X(c)$. Be sure to specify it for all values of c.

   (b) (6 points) Determine $\mathrm{P}\{X > 1\}$.

   (c) (6 points) Determine $\mathrm{P}\{X > 1 \mid X \leq 2\}$.

3. **[20 points]** Suppose T is a random variable with CDF $F_T(t) = \begin{cases} 1 - e^{-t^2} & t \geq 0 \\ 0 & \text{else} \end{cases}$.

   (a) (8 points) Is the distribution of T memoryless? If you answer *yes*, provide a proof. If you answer *no*, provide an example showing that T is not memoryless. (In either case, your answer should be based on the definition of memoryless.)

   (b) (6 points) Now consider the random variable $X = T^2$. Find the CDF and pdf of X.

   (c) (6 points) Is the distribution of X memoryless? If you answer *yes*, provide a proof. If you answer *no*, provide an example showing that X is not memoryless.

4. **[24 points]** Suppose you're signed up to receive news headlines via email to your cellphone, which arrive according to a Poisson process with rate 5 emails/hour.

   (a) (6 points) How long do you expect to wait between the first and third emails?

   (b) (6 points) You turn your cell phone off during the 50 minute lecture. How many emails do you expect to have arrived during that time?

   (c) (6 points) You turn your cell phone off during the 50 minute lecture. What is the probability that you receive 5 emails during that time?

   (d) (6 points) Suppose a hacker is sending you fake news emails according to a Poisson process with rate 10 emails/hour, independent of the Poisson process of the actual news headlines. Obtain $P\{\text{first email you receive is fake}\}$.

5. **[24 points]** Suppose X has the Gaussian distribution with mean 20 and variance 16 (i.e. the $N(20, 16)$ distribution).

   (a) (6 points) Find the maximum value of $f_X(u)$ over all values of u.

(b) (6 points) For what values of u is $f_X(u)$ equal to one half the maximum value of f_X? (Hint: $\exp(-\ln 2) = 0.5$ and $\ln 2 \approx 0.7$.

(c) (6 points) Carefully sketch the pdf, $f_X(u)$.

(d) (6 points) Express $P\{|X| \geq 30\}$ in terms of the Q function.

6. **[12 points]** Suppose X has the pdf: $f_\theta(u) = \begin{cases} \left(\frac{u}{\theta}\right) e^{-\frac{u^2}{2\theta}} & u \geq 0 \\ 0 & u < 0 \end{cases}$ with parameter $\theta > 0$. Suppose the value of θ is unknown and it is observed that $X = 10$. Find the maximum likelihood estimate, $\widehat{\theta}_{ML}$, of θ.

7. **[16 points]** Suppose (X, Y) is uniformly distributed over the set $\{(u, v) : 0 \leq v \leq u \leq 1\}$.

   (a) (8 points) Find $E[Y^2|X = u]$ for $0 \leq u \leq 1$.

   (b) (8 points) Find $\widehat{E}[Y|X = u]$. (Hint: There is a Y^2 in part (a) but only a Y in this part. This problem can be done with little to no calculation.)

8. **[24 points]** Suppose X and Y are jointly Gaussian with $\mu_X = 1$, $\mu_Y = 0$, $\sigma_X^2 = 1$, $\sigma_Y^2 = 4$, and $\rho_{XY} = \frac{1}{8}$.

   (a) (6 points) Let $W = X + \alpha Y + \beta$. Find the values of α and β that make X and W independent.

   (b) (6 points) Let $Z = 4X + 2Y + 3$. Find the mean and variance of Z.

   (c) (6 points) Find $E[Y|Z = 11]$.

   (d) (6 points) Find $E[Y^2|Z = 11]$.

9. **[15 points]** Suppose X and Y have the joint pdf:

$$f_{XY}(u, v) = \begin{cases} \frac{1}{2}\, e^{-u}, & u > 0,\ |v| < u \\ 0, & \text{elsewhere} \end{cases}$$

   (a) (9 points) Find $f_Y(v)$. (Be sure to find it for all values of v.)

   (b) (6 points) Find $E\left[\frac{1}{X}\right]$.

10. **[20 points]** Suppose that random variables X and Y have the joint probability density function (pdf):

$$f_{XY}(u, v) = \begin{cases} 2, & 0 < u \leq v < 1 \\ 0, & \text{elsewhere} \end{cases}$$

   (a) (8 points) Find the unconstrained minimum mean square error estimator of Y given $X = u$ for $0 \leq u \leq 1$.

   (b) (6 points) Find the minimum mean square error for the unconstrained estimate of Y given $X = u$.

   (c) (6 points) Find the best linear estimator of Y given $X = u$.

11. **[30 points]** (3 points per answer)
In order to discourage guessing, 3 points will be deducted for each incorrect answer (no penalty or gain for blank answers). A net negative score will reduce your total exam score.

(a) A and B are events with $0 < P(A) < 1$, $0 < P(B) < 1$, and $P(AB) > 0$.

TRUE	FALSE	
☐	☐	$P(A\|B^c) + P(A\|B) = 1$
☐	☐	If $P(A\|B) = 2P(A) = 1$, then A and B are independent
☐	☐	If $P(A) < P(B)$, then $P(A\|B) < P(B\|A)$

(b) Let X and Y be two continuous-type random variables such that f_X is symmetric around zero and $E[X^3]$ is well defined.

TRUE	FALSE	
☐	☐	$E[X^3] = 0$
☐	☐	$F_Y(v_1) < F_Y(v_2)$ for all v_1, v_2 such that $v_1 < v_2$
☐	☐	$P\{X > a, Y \leq b\} = (1 - F_X(a))\, F_Y(b)$

(c) Suppose X and Y are independent random variables, each uniformly distributed on the interval $[0, 1]$.

TRUE	FALSE	
☐	☐	$\mathrm{Cov}(X + Y, X^2) > 0$
☐	☐	$\mathrm{Var}(X + Y) = \frac{1}{6}$

(d) Suppose X has the binomial distribution with parameters $n = 20$ and $p = 0.2$ and variance σ_X^2.

TRUE	FALSE	
☐	☐	$P\{X = 4\} \geq P\{X = 5\}$.
☐	☐	$\sigma_X > 4$