University of Illinois Fall 2012

ECE 313: Hour Exam I

Monday October 8, 2012
7:00 p.m. — 8:15 p.m.
269 Everitt, 103 Talbot, & 1320 DCL

1. (a) P{at least 2 with same birth-month} = 1 - P{no 2 with same birth-month}
$= 1$ - $(1)(11/12)(10/12) = \frac{144-110}{144} = \frac{17}{72} \approx 0.24$

   (b) $P\{X=0\}$ = P{of players 1,2: 1 has the higher number } = 1/2
$P\{X=1\}$ = P{of players 1,2,3: 3 has largest, 1 the next largest} = (1/3)(1/2) = 1/6
$P\{X=2\}$ = P{of 1,2,3,4: 4 has largest, 1 the next largest} = (1/4)(1/3) = 1/12

   (c) $E[(2+Y)^2] = \text{Var}(2+Y) + (E[2+Y])^2 = \text{Var}(Y) + 9 = 14$
$\text{Var}(4+3Y) = 9\ \text{Var}(Y) = 45$

2. (a) In this case, knowing the color of one phone says nothing about the color of the other phone because the color choices are independent, therefore,
$P\{\text{both green}|\text{first phone is green}\} = P\{\text{green}\} = 1/3$.

   (b) Let G be the number of green phones bought. By the independence between the colors of the phones, $G \sim Binomial(2, 1/3)$. Therefore, using the definition of conditional probability,

$$P\{G=2|G\geq 1\} = \frac{P\{G=2, G\geq 1\}}{P\{G\geq 1\}} = \frac{P\{G=2\}}{1-P\{G=0\}} = \frac{(1/3)^2}{1-(2/3)^2} = 1/5.$$

   For the remainder of this problem, suppose you buy five phones, and let G be the number of green phones and B be the number of blue phones you buy.

   (c) By the independence between the colors of the phones, $G \sim Binomial(5, 1/3)$. Therefore, $E[G] = np = 5/3$ and $\text{Var}(G) = np(1-p) = 10/9$.

   (d) The events $\{G=3\}$ and $\{B=2\}$ are the same event because there are only five phones and each phone is either green or blue. An event cannot be independent of itself, unless it's the null set or the sample space, hence the events $\{G=3\}$ and $\{B=2\}$ are not independent. That is, $P\{G=3, B=2\} = P\{G=3\}$ and $P\{G=3\}P\{B=2\} = P\{G=3\}^2$, so $P\{G=3, B=2\} \neq P\{G=3\}P\{B=2\}$.

   (e) The event $\{B=G\}$ is the null set because there's an odd number of phones and each phone is either green or blue so the number of green phones cannot equal the number of blue phones. The null set is independent of any event, so the events $\{G=3\}$ and $\{B=G\}$ are independent. That is, $P\{G=3, B=G\} = P\{G=3\}P\{B=G\}$ because both sides of this equation are zero.

3. (a) The event $\{X=5\}$ happens if and only if the second, third, and fourth rolls all yield new numbers, and the fifth roll is equal to one of the first four numbers rolled. So $P\{X=5\} = \frac{5}{6}\frac{4}{6}\frac{3}{6}\frac{4}{6} = \frac{5}{27}$.

   (b) Given $X > 5$, i.e. given the first five rolls are distinct, the event $\{X=7\}$ happens if and only if the sixth roll is different from all of the first five, which has conditional probability $\frac{1}{6}$. Thus, $P(X=7|X>5) = \frac{1}{6}$.

4. (a) $P\{X=6\} = \frac{\lambda^6 e^{-\lambda}}{6!}$, where $\lambda = a^2+1$. The probability $P\{X=6\}$ is maximized with respect to λ by $\lambda = 6$ (by Example 2.8.4 in the notes.) This corresponds to $\widehat{a}_{ML} = \sqrt{5}$.

(b) $P\{X=0\} = e^{-\lambda} = e^{-(1+a^2)}$. So the problem is to select a to maximize the likelihood $e^{-(1+a^2)}$. Equivalently, the problem is to choose a to minimize $1+a^2$. That is done by $\widehat{a}_{ML} = 0$.

5. (a) Using the Law of Total Probability,

$$P(A) = P(A|S_1)P(S_1) + P(A|S_2)P(S_2) + P(A|S_3)P(S_3)$$

In state S_1, the source cannot produce two 1's, and therefore $P(A|S_1) = 0$. In S_2, a total of $\binom{4}{0} + \binom{4}{1} + \binom{4}{2} = 1+4+6 = 11$ sequences can be produced, and therefore $P(A|S_2) = \frac{1}{11}$. Similarly, in S_3, a total of $1+4+6+4 = 15$ sequences can be produced, and therefore $P(A|S_3) = \frac{1}{15}$. Thus

$$P(A) = \frac{1}{11}\frac{1}{3} + \frac{1}{15}\frac{1}{3} = \frac{26}{3 \times 11 \times 15} = \frac{26}{495}.$$

(b) By Bayes rule,

$$P(S_2|A) = \frac{P(A|S_2)P(S_2)}{P(A)} = \frac{\frac{1}{11}\frac{1}{3}}{\frac{1}{11}\frac{1}{3} + \frac{1}{15}\frac{1}{3}} = \frac{15}{26}.$$

6. (a) The likelihood ratio $\Lambda(k) = \frac{p_1(k)}{p_0(k)} = \frac{2k}{5}$ and therefore the ML rule is to declare H_1 whenever $\Lambda(X) \geq 1$, or equivalently, $X \geq 2.5$, which due to the integer constraints is equivalent to $X \geq 3$.

(b)

$$p_{\text{false-alarm}} = P(X \geq 1|H_0) = \sum_{k=3}^{4} \frac{1}{4} = \frac{1}{2}$$

and

$$p_{\text{miss}} = P(X \leq 2|H_1) = \frac{1}{10} + \frac{2}{10} = \frac{3}{10}$$

(c) The MAP rule is to declare H_1 whenever $\Lambda(X) \geq \frac{\pi_0}{\pi_1} = \frac{1}{2}$, or equivalently, $X \geq \frac{5}{4}$, which due to the integer constraints is equivalent to $X \geq 2$.