**University of Illinois** Fall 2012

# ECE 313: Hour Exam I

Monday October 8, 2012
7:00 p.m. — 8:15 p.m.
269 Everitt, 103 Talbot, & 1320 DCL

1. **[24 points]** This question has three separate parts:

   (a) (7 points) There are 3 persons in a room. Find the probability that *at least* 2 persons have the same *birth-month*. (For simplicity, assume that each of the twelve possibilities for the birth-month of a given person are equally likely.)

   (b) (9 points) Five distinct numbers are randomly distributed to players numbered 1 through 5. Whenever two players compare their numbers, the one with the higher number is declared the winner. Initially, players 1 and 2 compare their numbers; the winner then compares with player 3, and so on. Let $X$ denote the number of times player 1 is a winner. Find $P\{X = 0\}$, $P\{X = 1\}$, and $P\{X = 2\}$. (Hint: Your answers should sum to $\frac{3}{4}$.)

   (c) (8 points) If $E[Y]= 1$ and $\text{Var}(Y) = 5$, find $E[(2 + Y)^2]$, and find $\text{Var}(4 + 3Y)$.

2. **[21 points]** Suppose an on-line retailer sells Android phones at a bargain price but you can't choose the colors of the phones you buy. The color of each phone is green with probability 1/3; otherwise it is blue. Assume the colors of phones bought are mutually independent.

   (a) (5 points) For this part of the problem only, suppose you buy two phones. Given that the first phone you get out of the shipped box is green, find the conditional probability that both phones are green.

   (b) (5 points) For this part of the problem only, suppose you buy two phones and that you know that at least one of the phones is green. Find the conditional probability that both phones are green.

   *For the remainder of this problem, suppose you buy five phones, and let $G$ be the number of green phones and $B$ be the number of blue phones you buy.*

   (c) (5 points) Find $E[G]$ and $\text{Var}(G)$. (Express your answers as fractions.)

   (d) (3 points) Are the events $\{G = 3\}$ and $\{B = 2\}$ independent? Why are they or why are they not?

   (e) (3 points) Are the events $\{G = 3\}$ and $\{B = G\}$ independent? Why are they or why are they not?

3. **[14 points]** A fair six-sided die is rolled repeatedly. Each time the die is rolled, the number showing is written down. Let $X$ be the number of rolls until some number shows twice. The possible values of $X$ are 2,3,4,5,6, or 7. For each of the following parts, explain your reasoning and express your answer as a fraction in reduced form.

   (a) (7 points) Find $P\{X = 5\}$.

   (b) (1.8 points) Find $P\{X = 7|X > 5\}$.

4. **[12 points]** The number of photons $X$ detected by a particular sensor over a particular time period is assumed to have the Poisson distribution with mean $1+a^2$, where $a$ is the amplitude of an incident field. It is assumed $a \geq 0$, but otherwise $a$ is unknown.

   (a) (6 points) Find the maximum likelihood estimate, $\widehat{a}_{ML}$, of $a$ for the observation $X = 6$.

   (b) (6 points) Find the maximum likelihood estimate, $\widehat{a}_{ML}$, of $a$ given that it is observed $X = 0$.

5. **[14 points]** A message source that produces a sequence of 4 bits of information is equally likely to be in one of three states: $S_1$, $S_2$, and $S_3$. In state $S_k$, the 4 bit sequence can have at most $k$ 1's, with all such 4 bit sequences being equally likely. For example, in state $S_1$, the source can produce the sequences $0000, 1000, 0100, 0010, 0001$ with each of these sequences being produced with probability $\frac{1}{5}$.

   Let $A$ be the event that the sequence 0110 was produced by the source.

   (a) (7 points) Find $P(A)$.

   (b) (7 points) Find the conditional probability the source was in state $S_2$ given event $A$.

6. **[15 points]** Consider the hypothesis testing problem in which the pmf's of the observation $X$ under hypotheses $H_0$ and $H_1$ are given, respectively, by:

$$p_0(k) = \frac{1}{4} \text{ for } k = 1, 2, 3, 4.$$

   and

$$p_1(k) = \frac{k}{10} \text{ for } k = 1, 2, 3, 4.$$

   (a) (5 points) Find the ML decision rule, expressing your answer in terms of $X$ in the simplest possible way.

   (b) (5 points) Find $p_{\text{false-alarm}}$ and $p_{\text{miss}}$ for the ML decision rule.

   (c) (5 points) Find the MAP decision rule, expressing your answer in terms of $X$ in the simplest possible way, using priors $\pi_0 = \frac{1}{3}$ and $\pi_1 = \frac{2}{3}$ (You do *not* need to find the error probabilities.)