

ECE 313: Problem Set 12

Joint pdfs, Covariance, Minimum MSE estimation

Due: Wednesday November 30 at 4 p.m.

Reading: 313 Course Notes Sections 4.7 - 4.10

1. **[Joint pdfs of functions of random variables]**

Suppose X and Y have joint pdf $f_{X,Y}$, and $W = 2X - Y$ and $Z = X + 3Y$. Express the joint pdf of W and Z in terms of $f_{X,Y}$.

2. **[Covariance I]**

Consider random variables X and Y on the same probability space.

(a) If $\text{Var}(X + 2Y) = 40$ and $\text{Var}(X - 2Y) = 20$, what is $\text{Cov}(X, Y)$?

(b) In part (a), determine $\rho_{X,Y}$ if $\text{Var}(X) = 2 \cdot \text{Var}(Y)$.

The next two parts are independent of parts (a) and (b), and of each other. In particular, the numbers from part (a) are not to be assumed.

(c) If $\text{Var}(X + 2Y) = \text{Var}(X - 2Y)$, are X and Y uncorrelated?

(d) If $\text{Var}(X) = \text{Var}(Y)$, are X and Y uncorrelated?

3. **[Covariance II]**

Rewrite the expressions below in terms of $\text{Var}(X)$, $\text{Var}(Y)$, $\text{Var}(Z)$, and $\text{Cov}(X, Y)$.

(a) $\text{Cov}(3X + 2, 5Y - 1)$

(b) $\text{Cov}(2X + 1, X + 5Y - 1)$.

(c) $\text{Cov}(2X + 3Z, Y + 2Z)$ where Z is uncorrelated to both X and Y .

4. **[Linear minimum MSE estimation from uncorrelated observations]**

Suppose Y is estimated by a linear estimator, $L(X_1, X_2) = a + bX_1 + cX_2$, such that X_1 and X_2 have mean zero and are uncorrelated with each other.

(a) Determine a , b and c to minimize the MSE, $E[(Y - (a + bX_1 + cX_2))^2]$. Express your answer in terms of $E[Y]$, the variances of X_1 and X_2 , and the covariances $\text{Cov}(Y, X_1)$ and $\text{Cov}(Y, X_2)$.

(b) Express the MSE for the estimator found in part (a) in terms of the variances of X_1 , X_2 , and Y and the covariances $\text{Cov}(Y, X_1)$ and $\text{Cov}(Y, X_2)$.

5. **[What's new? or the innovations method]**

The previous problem shows that if Y is to be estimated by a linear combination of X_1 and X_2 , the solution is particularly simple if $\text{Cov}(X_1, X_2) = 0$. Intuitively, that is because, even if X_1 is already known, X_2 offers completely new information. If $\text{Cov}(X_1, X_2) \neq 0$, then by preprocessing, we can reduce the problem to the previous case. The idea is to let $\widetilde{X}_2 = X_2 - hX_1$, where the constant h is chosen so that X_1 and \widetilde{X}_2 are uncorrelated. Intuitively, \widetilde{X}_2 is the part of X_2 that is new, or innovative, to someone who already knows X_1 . Then, the best linear estimator of Y given X_1 and \widetilde{X}_2 can be easily found by the method of the previous problem. But a linear estimator based on X_1 and \widetilde{X}_2 can also be expressed as a linear estimator based on X_1 and X_2 , or vice versa. So the estimator would also be the best linear estimator of Y given X_1 and X_2 .

This recipe is illustrated by an example as follows. Suppose Y , X_1 , and X_2 have mean zero and variance one, and suppose $\text{Cov}(Y, X_1) = \text{Cov}(Y, X_2) = 0.8$ and $\text{Cov}(X_1, X_2) = 0.5$.

(a) Find h so that X_1 and $\widetilde{X}_2 = X_2 - hX_1$ are uncorrelated.

- (b) Find $\text{Var}(\tilde{X}_2)$ and $\text{Cov}(Y, \tilde{X}_2)$.
- (c) Appealing to the previous problem, find the values of a , b , and c so that the MSE for estimating Y by the linear estimator $a + bX_1 + c\tilde{X}_2$ is minimized, and find the resulting MSE.
- (d) Express the linear estimator found in part (c) as a linear combination of X_1 and X_2 .

6. **[An estimation problem]**

Suppose X and Y have the following joint pdf:

$$f_{X,Y}(u,v) = \begin{cases} \frac{8uv}{(15)^4} & u \geq 0, v \geq 0, u^2 + v^2 \leq (15)^2 \\ 0 & \text{else} \end{cases}$$

- (a) Find the constant estimator, δ^* , of Y , with the smallest mean square error (MSE), and find the MSE.
- (b) Find the unconstrained estimator, $g^*(X)$, of Y based on observing X , with the smallest MSE, and find the MSE.
- (c) Find the linear estimator, $L^*(X)$, of Y based on observing X , with the smallest MSE, and find the MSE. (Hint: You may use the fact $E[XY] = \frac{75\pi}{4} \approx 58.904$, which can be derived using integration in polar coordinates.)