University of Illinois **Fall 2011**

ECE 313: Problem Set 9
Functions of a Random Variable, Maximum Likelihood Estimation, Hypothesis Testing

Due: Wednesday, October 26 at 4 p.m.
Reading: *ECE 313 Course Notes,* Sections 3.7-3.9.

1. []
The current I through a semiconductor diode is related to the voltage V across the diode as $I = I_0(\exp(V) - 1)$ where I_0 is the magnitude of the reverse current. Suppose that the voltage across the diode is modeled as a continuous random variable $\mathcal{V}$ with pdf

$$f_{\mathcal{V}}(u) = 0.5\exp(-|u|), -\infty < u < \infty.$$

Then, the current $\mathcal{I}$ is also a continuous random variable.

   (a) What values can $\mathcal{I}$ take on?

   (b) Find the CDF of $\mathcal{I}$.

   (c) Find the pdf of $\mathcal{I}$.

2. []
A signal $\mathbb{X}$ is modeled as a unit (standard) Gaussian random variable. For some applications, however, only the quantized value $\mathbb{Y}$ (where $\mathbb{Y} = \alpha$ if $\mathbb{X} > 0$ and $\mathbb{Y} = -\alpha$ if $\mathbb{X} \leq 0$) is used. Note that $\mathbb{Y}$ is a *discrete* random variable.

   (a) What is the pmf of $\mathbb{Y}$?

   (b) The *squared error* in representing $\mathbb{X}$ by $\mathbb{Y}$ is $\mathbb{Z} = \begin{cases} (\mathbb{X}-\alpha)^2, & \text{if } \mathbb{X} > 0, \\ (\mathbb{X}+\alpha)^2, & \text{if } \mathbb{X} \leq 0, \end{cases}$ and varies as different trials of the experiment produce different values of $\mathbb{X}$. We would like to choose the value of α so as to minimize the *mean* squared error $\mathsf{E}[\mathbb{Z}]$. Use LOTUS to calculate $\mathsf{E}[\mathbb{Z}]$ (the answer will be a function of α), and then find the value of α that minimizes $\mathsf{E}[\mathbb{Z}]$.
   The choice of α that minimizes $\mathsf{E}[\mathbb{Z}]$ is $\frac{2}{\sqrt{2\pi}}$.

   (c) We now get more ambitious and use a 3-bit A/D converter which first quantizes $\mathbb{X}$ to the nearest integer $\mathbb{W}$ in the range 3 to +3. Thus, $\mathbb{W} = 3$ if $\mathbb{X} \geq 2.5$, $\mathbb{W} = 2$ if $1.5 \leq \mathbb{X} < 2.5$, $\mathbb{W} = 1$ if $0.5 \leq \mathbb{X} < 1.5$, $\cdots$, $\mathbb{W} = -3$ if $\mathbb{X} < -2.5$. Note that $\mathbb{W}$ is also a discrete random variable. Find the pmf of $\mathbb{W}$.

3. []
Consider the following binary hypothesis testing problem. If hypothesis H_0 is true, the continuous random variable $\mathbb{X}$ is uniformly distributed on the interval $(-2, 2)$, while if hypothesis H_1 is true, the pdf of $\mathbb{X}$ is $f_1(u) = \begin{cases} \frac{1}{4}(2 - |u|), & |u| < 2, \\ 0, & \text{otherwise.} \end{cases}$

   (a) The *maximum-likelihood* decision rule can be stated in the form $|\mathbb{X}| \underset{\mathsf{H}_{1-x}}{\overset{\mathsf{H}_x}{\gtrless}} \eta$. Specify whether x denotes 0 or 1, and find the values of η, the probability of false alarm P_{FA}, and the probability of missed detection P_{MD}.

   (b) Suppose the hypotheses have *a priori* probabilities $\pi_0 = 1/3$ and $\pi_1 = 2/3$. What is the error probability $P(E)$ of the maximum-likelihood decision rule?

(c) The MAP (also known as the minimum-error-probability or Bayesian) decision rule can be stated in the form $|\mathbb{X}| \underset{\mathsf{H}_{1-x}}{\overset{\mathsf{H}_x}{\gtrless}} \xi$. Specify whether x denotes 0 or 1, and find the values of ξ and the error probability $P(E)$.

4. []
Consider a sphere whose radius is a random variable $\mathcal{R}$ with pdf $f_{\mathcal{R}}(u) = 2u,\ 0 < u < 1$, and 0 otherwise.

(a) What is the average radius of the sphere?

(b) What is the average volume?

(c) What is the average surface area?

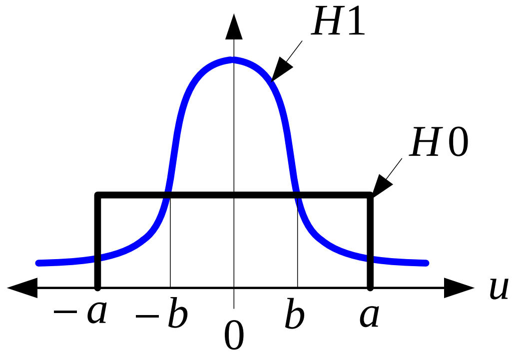
(d) If a sphere of average radius is called an *average sphere*, then does an average sphere have the average volume? Does it have the average surface area?

5. []
$\mathbb{X}$ denotes a *uniform* random variable with mean 1 and variance 3. Find the pdf of $\mathbb{Y} = |\mathbb{X}|$.

6. []
An observation X is drawn from a standard normal distribution (i.e. $N(0,1)$) if hypothesis H_1 is true and from a uniform distribution with support $[-a, a]$ if hypothesis H_0 is true. As shown in the figure below (under part (b)), the pdfs of the two distributions are equal when $|u| = b$.

(a) Express the maximum likelihood (ML) decision rule in a simple way, in terms of the observation X and the constants a and b.

(b) Shade and label the regions in the figure below such that the area of one of the regions is $p_{false\ alarm}$ and the area of the other region is p_{miss}.



(c) Express $p_{false\ alarm}$ and p_{miss} for the ML decision rule in terms of the constants a, b, and the Φ function or Q function with positive arguments.

(d) **[6 points]** Determine the maximum *á posteriori* probability (MAP) rule when $a = \frac{3}{2}$, $b = 0.6$, and the probability of hypothesis $H1$ being true is. $\pi_1 = \frac{\sqrt{2\pi}}{3+\sqrt{2\pi}}$.