University of Illinois **Fall 2011**

ECE 313: Problem Set 6
Decision Making Under Uncertainty, Reliability

Due: Wednesday, October 5 at 4 p.m.
Reading: *ECE 313 Course Notes,* Sections 2.11-2.12.

1. **[Detection problem with a geometric model]**
A transmitter chooses one of two routes (Route 0 or Route 1) and repeatedly transmits a packet over the chosen route until the packet is received without error (that is, without CRC checksum failure) at the receiver. $\mathbb{X}$ denotes the number of times the packet is transmitted over the chosen route including the final error-free transmission. Assuming that the successive transmissions are independent trials of an experiment, the two hypotheses are

   - H_1: Route 1 is used for packet transmission: $\mathbb{X} \sim \mathsf{Geometric}(p_1)$
   - H_0: Route 0 is used for packet transmission: $\mathbb{X} \sim \mathsf{Geometric}(p_0)$

   where $0 < p_1 < p_0 < 1$ are the probabilities of error-free transmission over the two routes.

   (a) State the maximum-likelihood decision rule as to which route was used as a threshold test on the observed value of $\mathbb{X}$.

   (b) Suppose the transmitter chooses Route 0 and Route 1 with probabilities π_0 and $\pi_1 = 1 - \pi_0$ respectively, i.e., π_0 and π_1 are the *a priori* probabilities of hypotheses H_0 and H_1. Assume that $0 < \pi_0 < 1$.
   For what values of π_0 (if any) does the minimum-error-probability decision rule always choose hypothesis H_1 regardless of the value of the observation $\mathbb{X}$?
   For what values of π_0 (if any) does the MAP decision rule always choose hypothesis H_0 regardless of the value of the observation $\mathbb{X}$?

2. []
$\mathsf{H}_0, \mathsf{H}_1$, and H_2 respectively denote the hypotheses that a student is excellent, good, or average (there are no poor students). The number of grade points earned by the student in a course is a random variable $\mathbb{X}$ that takes on values 3, 6, 9, and 12 only. The professor knows that the pmf of $\mathbb{X}$ when H_0 is true is $p_0(12) = 0.75, p_0(9) = 0.15, p_0(6) = 0.07, p_0(3) = 0.03$, that is, an excellent student has 75% chance of doing well enough on the exam to get an A, 15% chance of a B, etc. Similarly, when H_1 is the true hypothesis, the pmf of $\mathbb{X}$ is $p_1(12) = 0.15, p_1(9) = 0.6, p_1(6) = 0.15, p_1(3) = 0.1$, while if H_2 is true, $p_2(12) = 0.05, p_2(9) = 0.1, p_2(6) = 0.65, p_2(3) = 0.2$. The professor observes $\mathbb{X}$ and must decide which of the hypotheses $\mathsf{H}_0, \mathsf{H}_1, \mathsf{H}_2$ is true.

   (a) What is the professor's maximum-likelihood decision rule?

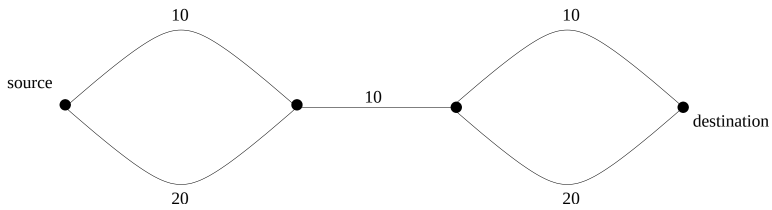
   (b) What is the probability that an excellent student is mistakenly labeled as good? What is the probability that an excellent student is mistakenly labeled as average? What is the probability that an average student is classified either as good or as excellent?

   (c) If $P(\mathsf{H}_0) = 0.2, P(\mathsf{H}_1) = 0.55$, and $P(\mathsf{H}_2) = 0.25$, what is the probability that the maximum-likelihood decision rule mis-classifies students?

   (d) What is the MAP decision rule corresponding to these probabilities and what is the probability that the MAP decision rule mis-classifies students?

3. []
Let A and B denote two *mutually exclusive* events that can occur on a trial of an experiment. Repeated independent trials of the experiment are carried out until either the event A or the event B occurs. What is the probability that A occurs before B does?

4. **[Network capacity and failure]**
Consider the following network, with link capacities (in packets) as shown. Assume that each link fails independently with probability $p = 1/2$.

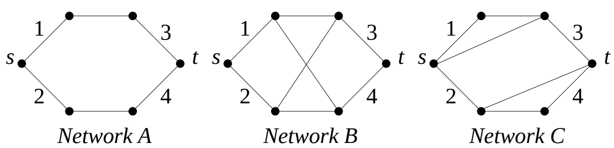


(a) What is the probability that a message can be sent successfully from the *source* to the *destination*?

(b) Let $\mathbb{X}$ denote the number of packets that can be sent from the *source* to the *destination*. Determine the pmf for $\mathbb{X}$.

(c) Determine the expected value of $\mathbb{X}$.

5. **[Reliability of three $s - t$ networks]**
Consider $s - t$ networks A through C shown.



For each network, suppose that each link fails with probability p, independently of the other links, and suppose network outage occurs if at least one link fails on each path from s to t. Not all the links are labeled–for example, Network A has six links. For simplicity, we suppose that links can only be used in the forward direction, so that the paths with five links for Network B do not count;. Let F denote the event of network outage.

(a) Find $P(F)$ in terms of p for Network A, and give the numerical value of $P(F)$ for $p = 0.001$, accurate to within four significant digits.

(b) This part aims to find $P(F)$ for Network B, using the law of total probability, based on the partition of the sample space into the events: $D_0, D_1, D_{2,s}, D_{2,d}, D_3, D_4$. Here, D_i is the event that exactly i links from among links $\{1, 2, 3, 4\}$ fail, for $i = 0, 1, 3, 4$; $D_{2,s}$ is the event that exactly two links from among links $\{1, 2, 3, 4\}$ fail and they are on the same side (i.e. either links 1 and 2 fail or links 3 and 4 fail); $D_{2,d}$ is the event that exactly two links from among links $\{1, 2, 3, 4\}$ fail and they are on different sides. Find the probability of each of these events, find the conditional probabilities of F given any one of these events, and finally, find $P(F)$. Express your answers as a function of p, and give the numerical value of $P(F)$ for $p = 0.01$, accurate to within four significant digits.

(c) Find the numerical value of $P(D_{2,s}|F)$ for $p = 0.001$.