University of Illinois **Fall 2011**

ECE 313: Problem Set 3
Conditional probabilities, independence, and the binomial distribution

Due: Wednesday September 14 at 4 p.m.
Reading: *313 Course Notes* Sections 2.3–2.4

1. **[Conditional probability and independence]**
There are three dice in a bag (each die has six faces). One of the dice one red face, another has two red faces, and the third has three red faces. One of the dice is drawn at random from the bag, each die having an equal chance of being drawn. The selected die is rolled repeatedly.
   (a) What is the probability that red comes up on the first roll?
   (b) Given that red comes up on the first roll, what is the conditional probability that red comes up on the second roll?

2. **[Conditional probability and independence]**
In a game of Risk, an attack is simulated by two players rolling dice. Player 1 rolls three white dice, and Player 2 rolls two red dice. Define the random variables W_1 and W_2 to be the largest and second largest numbers showing on the the white dice, respectively. Note that W_1 could be equal to W_2. Define R_1 and R_2 to be the higher and lower numbers shown on the red dice, respectively. Note that R_1 could be equal to R_2. For example, if Player 1 rolls 4, 2, 3 we have $W_1 = 4$, $W_2 = 3$, while if Player 1 rolls 4, 2, 4 we have $W_1 = 4$, $W_2 = 4$.
   (a) Determine the pmf for R_1.
   (b) Determine the pmf for R_2.
   (c) Determine the pmf for W_1.
   (d) Are the events $\{R_1 = 5\}$ and $\{R_2 = 4\}$ independent? Justify (mathematically) your answer.
   (e) Are the events $\{R_1 = 5\}$ and $\{W_2 = 2\}$ independent? Justify (mathematically) your answer.
   (f) Compute the conditional probability $P\{R_2 = 2|R_1 = 3\}$.

3. **[Independence]**
A gambling book recommends the following winning strategy for a game of roulette, where \$1 bets are made on each spin of the roulette, getting back \$2 in total (the \$1 bet plus \$1 in winnings) if the gambler 'wins' the bet, and getting \$0 back (hence loosing \$1) if the gambler 'looses' the bet. The strategy recommends that a gambler always bet on red (the probability of red is 18/38) according to the next two rules depending on the outcome of the initial bet: if the gambler wins the initial bet, she should stop betting and call it the night; however if the gambler looses the initial bet, she should bet on each of the next two spins of the roulette and stop betting no matter their outcome. Let X denote the amount of money the gambler possesses at the end of the game, assuming that she started with \$3.
   (a) Find $P\{X > 3\}$.
   (b) Find $E[X]$.
   (c) Are you convinced that the strategy is indeed a winning strategy? Why or why not?

4. **[Conditional probability and the binomial distribution]**
Let X denote the number of heads observed in 10 tosses of a fair coin.
   (a) Find $P\{X \leq 5|X \geq 4\}$.
   (b) Given that $X = 4$, what is the (conditional) probability that the 4th toss was a head?

(c) You have a strong suspicion that the coin that was tossed is not a fair coin. Nonetheless, your friendly neighborhood bookmaker asks if you want to bet that the 4th toss was a head knowing only that the event $\{X = 4\}$ has occurred. The bookie knows the outcome of the 4th toss (but cannot change it after you have placed your bet!) as well as the value of $P(head) = p$. Does the fact that you don't know the value of p put you in a disadvantage? Why or why not?

5. **[Binomial distribution]**
A New Yorker runs an investment management service that has the stated goal of doubling the value of his clients investments in a week via day trading. His brochure boasts that, "On average, my clients triple their money in five weeks!" After poring over back issues of the Wall Street Journal you learn the truth: at the end of any week, the investments of his clients will have doubled with probability 0.5, and will have decreased by 50% with probability 0.5. Thus, at the end of the first week, an initial investment of \$32 will be worth either \$64 or \$16, each with probability 0.5. Performance in any week is independent of performance during the other weeks. Anxious to apply your new skills in probability theory, you decide to invest \$32, and to let that investment ride for five weeks (in fact, you decide not to even look at the stock prices until the five weeks are over). Let the random variable X denote the value in dollars of your investment at the end of a five week period.

   (a) What are the possible values of X?

   (b) What is the pmf of the random variable X?

   (c) What is the expected value of X? Is the TV commercial accurate?

   (d) What is the probability that you will lose money on your investment?