University of Illinois **Fall 2011**

ECE 313: Problem Set 12: Solutions
Joint pdfs, Covariance, Minimum MSE estimation

1. [**Joint pdfs of functions of random variables**]
Write $\binom{W}{Z} = A\binom{X}{Y}$, where

$$A = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix} \qquad det(A) = 7 \qquad A^{-1} = \frac{1}{7}\begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}.$$

That is, the linear transformation used here maps a point $\binom{u}{v}$ to $\binom{\alpha}{\beta}$ such that $\alpha = 2u - v$ and $\beta = u + 3v$. The inverse mapping is given by $u = \frac{3\alpha+\beta}{7}$ and $v = \frac{2\beta-\alpha}{7}$, or equivalently, $\binom{u}{v} = A^{-1}\binom{\alpha}{\beta}$. By proposition 4.7.1, we have

$$f_{W,Z}(\alpha, \beta) \;=\; \frac{1}{7}\, f_{X,Y}\, (\frac{3\alpha+\beta}{7}, \frac{2\beta-\alpha}{7}),$$

for all $(\alpha, \beta) \in \mathcal{R}^2$.

2. [**Covariance I**]

   (a)

$$\begin{aligned} \text{Var}(X+2Y) &= \text{Cov}(X+2Y, X+2Y) \\ &= \text{Var}(X) + 4\text{Var}(Y) + 4\text{Cov}(X,Y) = 40 \end{aligned}$$

   Similarly, $\text{Var}(X-2Y) = \text{Cov}(X-2Y, X-2Y) = \text{Var}(X)+4\text{Var}(Y)-4\text{Cov}(X,Y) = 20$. Taking the difference of the two equations describing $\text{Var}(X+2Y)$ and $\text{Var}(X-2Y)$yields $\text{Cov}(X,Y) = 2.5$.

   (b) Adding the two equations describing $\text{Var}(X+2Y)$ and $\text{Var}(X-2Y)$, we get

$$\begin{aligned} 2\text{Var}(X) + 8\text{Var}(Y) &= 60 \\ 12\text{Var}(Y) = 60 \end{aligned}$$

   Hence, $\text{Var}(Y) = 5$, $\text{Var}(X) = 10$, and

$$\rho_{X,Y} = \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}} = 0.3536$$

The next two parts are independent of parts (a) and (b), and of each other. In particular, the numbers from part (a) are not to be assumed.

   (c) From the expressions for $\text{Var}(X+2Y)$ and $\text{Var}(X-2Y)$ in terms of the variances and covariances in part (a), the condition $\text{Var}(X+2Y) = \text{Var}(X-2Y)$ implies that $\text{Cov}(X,Y) = 0$. Hence, X and Y are uncorrelated.

   (d) No. The condition $\text{Var}(X) = \text{Var}(Y)$ does not imply that $\text{Cov}(X,Y) = 0$.

3. [**Covariance II**]

   (a) $\text{Cov}(3X+2, 5Y-1) = \text{Cov}(3X, 5Y) = 15\text{Cov}(X,Y)$.

   (b)

$$\begin{aligned} \text{Cov}(2X+1, X+5Y-1) &= \text{Cov}(2X, X+5Y) = \text{Cov}(2X, X) + \text{Cov}(2X, 5Y) \\ &= 2\text{Cov}(X,X) + 10\text{Cov}(X,Y) = 2\text{Var}(X) + 10\text{Cov}(X,Y) \end{aligned}$$

(c)

$$\begin{aligned}\mathrm{Cov}(2X+3Z,Y+2Z) &= \mathrm{Cov}(2X,Y)+\mathrm{Cov}(2X,2Z)+\mathrm{Cov}(3Z,Y)+\mathrm{Cov}(3Z,2Z)\\ &= 2\mathrm{Cov}(X,Y)+4\mathrm{Cov}(X,Z)+3\mathrm{Cov}(Z,Y)+6\mathrm{Cov}(Z,Z)\\ &= 2\mathrm{Cov}(X,Y)+6\mathrm{Var}(Z)\end{aligned}$$

4. **[Linear minimum MSE estimation from uncorrelated observations]**

   (a) The MSE can be written as $E[((Y-bX_1-cX_2)-a)^2]$, which is the same as the MSE for estimation of $Y-bX_1-cX_2$ by the constant a. The optimal choice of a is $E[Y-bX_1-cX_2]=E[Y]$. Substituting $a=E[Y]$, the MSE satisfies

$$\begin{aligned}\mathrm{MSE} &= \mathrm{Var}(Y-bX_1-cX_2)\\ &= \mathrm{Cov}(Y-bX_1-cX_2,Y-bX_1-cX_2)\\ &= \mathrm{Cov}(Y,Y)+b^2\mathrm{Cov}(X_1,X_1)-2b\mathrm{Cov}(Y,X_1)+c^2\mathrm{Cov}(X_2,X_2)-2c\mathrm{Cov}(Y,X_2)\\ &= \mathrm{Var}(Y)+\big(b^2\mathrm{Var}(X_1)-2b\mathrm{Cov}(Y,X_1)\big)+\big(c^2\mathrm{Var}(X_2)-2c\mathrm{Cov}(Y,X_2)\big). \qquad (1)\end{aligned}$$

   The MSE is quadratic in b and c and the minimizers are easily found to be $b=\frac{\mathrm{Cov}(Y,X_1)}{\mathrm{Var}(X_1)}$ and $c=\frac{\mathrm{Cov}(Y,X_2)}{\mathrm{Var}(X_2)}$. Thus, $L(X_1,X_2)=E[Y]+\frac{\mathrm{Cov}(Y,X_1)}{\mathrm{Var}(X_1)}X_1+\frac{\mathrm{Cov}(Y,X_2)}{\mathrm{Var}(X_2)}X_2$.

   (b) Substituting the values of b and c found into (**??**) yields

$$\mathrm{MSE}=\mathrm{Var}(Y)-\frac{\mathrm{Cov}(Y,X_1)^2}{\mathrm{Var}(X_1)}-\frac{\mathrm{Cov}(Y,X_2)^2}{\mathrm{Var}(X_2)}.$$

5. **[What's new? or the innovations method]**

   (a) $\mathrm{Cov}(X_1,X_2-hX_1)=\mathrm{Cov}(X_1,X_2)-h\mathrm{Var}(X_1)=0.5-h$, so $h=0.5$. Thus, $\widetilde{X_2}=X_2-(0.5)X_1$.

   (b)

$$\begin{aligned}\mathrm{Var}(\widetilde{X}_2) &= \mathrm{Cov}(X_2-(0.5)X_1,X_2-(0.5)X_1)\\ &= \mathrm{Var}(X_2)-2(0.5)\mathrm{Cov}(X_1,X_2)+(0.5)^2\mathrm{Var}(X_1)\\ &= 1-0.5+0.25=0.75.\\ \mathrm{Cov}(Y,\widetilde{X}_2) &= \mathrm{Cov}(Y,X_2-(0.5)X_1)\\ &= \mathrm{Cov}(Y,X_2)-(0.5)\mathrm{Cov}(Y,X_1)\\ &= 0.8-(0.5)(0.8)=0.4\end{aligned}$$

   (c) $a=E[Y]=0$, $b=\frac{\mathrm{Cov}(Y,X_1)}{\mathrm{Var}(X_1)}=0.8$, and $c=\frac{\mathrm{Cov}(Y,\widetilde{X}_2)}{\mathrm{Var}(\widetilde{X}_2)}=\frac{0.4}{0.75}$, and

$$\mathrm{MSE}=\mathrm{Var}(Y)-\frac{\mathrm{Cov}(Y,X_1)^2}{\mathrm{Var}(X_1)}-\frac{\mathrm{Cov}(Y,\widetilde{X}_2)^2}{\mathrm{Var}(\widetilde{X}_2)}=1-0.64-\frac{(0.4)^2}{0.75}=0.1466\ldots.$$

   (d) The estimator is $L^*=(0.8)X_1+\frac{0.4}{0.75}(X_2-(0.5)X_1)=\frac{0.4}{0.75}(X_1+X_2)$.

6. **[An estimation problem]**

   (a) We know $\delta^*=E[Y]$, and the resulting MSE is $\mathrm{Var}(Y)$. We could directly compute the first and second moments of Y, but it is about the same amount work if f_Y is found first, so we find f_Y. The support of f_Y is [0,15]. For $0\le v\ \le 15$,

$$f_Y(v)=\int_0^{\sqrt{225-v^2}}\frac{8uv}{15^4}du=\frac{4u^2v}{15^4}\bigg|_{u=0}^{\sqrt{225-v^2}}=\frac{4v}{225}\left(1-\frac{v^2}{225}\right)$$

Thus,

$$\delta^* = E[Y] = \int_0^{15} \frac{4v^2}{225}\left(1 - \frac{v^2}{225}\right) dv = 8,$$

and

$$E[Y^2] = \int_0^{15} \frac{4v^3}{225}\left(1 - \frac{v^2}{225}\right) dv = 75,$$

so MSE(using δ^*)=Var$(Y) = 75 - 8^2 = 11$.

(b) We know $g^*(u) = E[Y|X = u]$. To compute g^* we thus need to find $f_{Y|X}(v|u)$. By symmetry, X and Y have the same distribution, so

$$f_X(u) = f_Y(u) = \begin{cases} \frac{4u}{225}\left(1 - \frac{u^2}{225}\right) & 0 \le u \le 15 \\ 0 & \text{else.} \end{cases}$$

Thus, $f_{Y|X}(v|u)$ is well defined for $0 \le u \le 15$. For such u,

$$f_{Y|X}(v|u) = \frac{f_{X,Y}(u,v)}{f_X(u)} = \begin{cases} \frac{2v}{225-u^2} & 0 \le v \le \sqrt{225-u^2} \\ 0 & \text{else.} \end{cases}$$

That is, for u fixed, the conditional pdf of Y has a triangular shape over the interval $\left[0, \sqrt{225-u^2}\right]$. Thus, for $0 \le u \le 15$,

$$g^*(u) = \int_0^{\sqrt{225-u^2}} \frac{2v^2}{225-u^2} dv = \frac{2\sqrt{225-u^2}}{3}.$$

To compute the MSE for g^* we find

$$E[g^*(X)^2] = \int_0^{15} g^*(u)^2 f_X(u) du = \int_0^{15} \frac{4(225-u^2)}{9} \frac{4u}{225}\left(1 - \frac{u^2}{225}\right) du = \frac{200}{3}.$$

Therefore, MSE(using g^*) $= E[Y^2] - E[g^*(X)^2] = \frac{25}{3} = 8.333\ldots$.

(c) Using the hint, $\mathrm{Cov}(X,Y) = E[XY] - E[X]E[Y] = \frac{75\pi}{4} - 64 \approx -5.0951$. Thus,

$$L^*(u) = E[Y] + \frac{\mathrm{Cov}(X,Y)}{\mathrm{Var}(X)}(u - E[X]) = 8 - (0.4632)(u - 8)$$

and

$$\text{MSE(using } L^*) = \mathrm{Var}(Y) - \frac{\mathrm{Cov}(X,Y)^2}{\mathrm{Var}(X)} = 8.6400$$