**University of Illinois** **Fall 2011**

# ECE 313: Problem Set 11: Solutions
# Distribution of sums of random variables and additional examples using joint distributions

1. **[Sums of two random variables]**

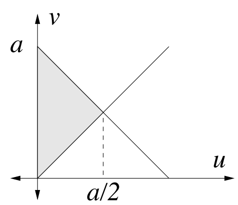
   (a) The joint pdf has to integrate to one, therefore
   $1 = \int_0^1 \int_1^5 \left(\frac{x}{5} + cy\right) dydx = \frac{2}{5} + 12c \quad \Rightarrow \quad c = \frac{1}{20}$

   (b) No, the density does not factor.

   (c) $P\{X + Y > 3\} = \int_0^1 \int_{3-x}^5 \left(\frac{x}{5} + \frac{y}{20}\right) dydx = \frac{11}{15}$

2. **[Sums of random variables]**

   The variable $Z$ takes values in the positive reals, and for $a \geq 0$, $F_Z(a) = P\{Z \leq a\}$ is equal to the integral of the joint pdf over the shaded region:




   The integral can be computed using either integration with respect to $u$ on the outside or integration with respect to $v$ on the outside. Using $u$ on the outside permits us to do the computation using only one double integral:

$$F_Z(a) = \int_0^{a/2} \int_u^{a-u} 2e^{-u-v} dv\ du = 1 - (1+a)e^{-a}.$$

   Therefore,

$$F_Z(a) = \begin{cases} 1 - (1+a)e^{-a} & a \geq 0 \\ 0 & \text{else.} \end{cases}$$

   Differentiating the CDF yields $f_Z(a) = ae^{-a}$ for $a \geq 0$. That is, $Z$ has the gamma distribution with parameters $r = 2$ and $\lambda = 1$. (This could have been deduced without calculation. The joint CDF is the same as the joint CDF of two independent exponential random variables, conditioned on the first to be smaller than the second. Since the two random variables are independent and identically distributed, such conditioning does not change the distribution of the sum, and the sum of two independent exponentials has the gamma distribution with $r = 2$.)

3. **[Joint densities]**

   (a) Let $T_i$ denote the time at which a shock of type $i$ occurs, for $i = 1, 2, 3$. Then, for $s > 0$ and $t > 0$,

$$\begin{aligned} P\{X_1 > s, X_2 > t\} &= P\{T_1 > s, T_2 > t, T_3 > \max(s,t)\} = P\{T_1 > s\}P\{T_2 > t\}P\{T_3 > \max(s,t)\} \\ &= e^{-\lambda_1 s}e^{-\lambda_2 t}e^{-\lambda_3 \max(s,t)} = e^{-(\lambda_1 s + \lambda_2 t + \lambda_3 \max(s,t))} \end{aligned}$$

   (b) If $s < 0$ then $F_{X_1}(s) = 0$, and for $s \geq 0$
   $F_{X_1}(s) = P\{X_1 \leq s\} = 1 - P\{X_1 > s\} = 1 - P\{X_1 > s, X_2 > 0\} = 1 - e^{-(\lambda_1 s + \lambda_3 s)} = 1 - e^{-s(\lambda_1 + \lambda_3)}$.
   So, $X_1$ is an exponential random variable with parameter $\lambda_1 + \lambda_3$.

4. **[Joint densities and functions of two random variables]**

(a)

$$\int\int_{\{u^2+v^2<1\}} A\left(1-\sqrt{u^2+v^2}\right) dudv = \int_0^{2\pi}\int_0^1 A(1-r)rdrd\theta = \frac{A\pi}{3} \to A = \frac{3}{\pi}$$

(b) The random variable $Z$ takes values in the interval $[0,1]$. For $0 \leq c \leq 1$,

$$\begin{aligned} F_Z(c) &= P\{Z \leq c\} = \int\int_{u^2+v^2<c} \frac{3}{\pi}\left(1-\sqrt{u^2+v^2}\right) dudv \\ &= \int_0^{2\pi}\int_0^{\sqrt{c}} \frac{3(1-r)}{\pi} rdrd\theta = 3c - 2c^{3/2}. \end{aligned}$$

Then pdf of $Z$ is

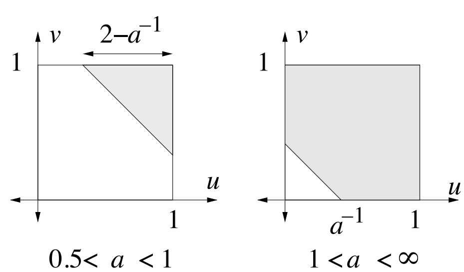
$$f_Z(c) = \begin{cases} 3\left(1-\sqrt{c}\right) & 0 \leq c \leq 1 \\ 0 & \text{else.} \end{cases}$$

(c)

$$\begin{aligned} E[Z^5] &= \int\int_{\{u^2+v^2<1\}} \left(u^2+v^2\right)^5 \frac{3\left(1-\sqrt{u^2+v^2}\right)}{\pi} dudv \\ &= \int_0^{2\pi}\int_0^1 r^{10}\frac{3(1-r)}{\pi} rdrd\theta \\ &= 6\int_0^1 (r^{10}-r^{11})dr = \frac{1}{22}. \end{aligned}$$

5. **[Joint densities and functions of two random variables]**

One approach to this problem is to use the fact that the distribution of $X+Y$ has the triangular pdf as found in Example 4.5.4 in the notes. Here we take a direct approach. Note that $Z$ takes values in the set $[\frac{1}{2}, \infty)$. For $a$ in that range, $F_Z(a) = P\{Z \leq a\} = P\{1/(X+Y) \leq a\} = P\{X+Y \geq a^{-1}\}$ Considering two cases separately, note that $P\{X+Y \geq a^{-1}\}$ is the area of the shaded region:




Therefore,

$$F_Z(a) = \begin{cases} 0 & a < 1/2 \\ (2-a^{-1})^2/2 & 1/2 \leq a \leq 1 \\ 1-a^{-2}/2, & 1 < a < \infty \end{cases}$$

$$f_Z(a) = \begin{cases} 2a^{-2} - a^{-3}, & 1/2 \leq a \leq 1 \\ a^{-3}, & 1 < a < \infty \end{cases}$$