ECE 313: Problem Set 10: Solutions
Joint Distributions, Independence

1. [**A joint pmf**]

(a) The marginal pmfs are the column and row sums shown in the table below.

	u=0	u=1	u=2	u=3	Row sum = $p_Y(v)$
v=4	0	0.1	0.1	0.2	0.4
v=5	0.2	0	0	0	0.2
v=6	0	0.2	0.1	0.1	0.4
Column sum $p_X(u)$	0.2	0.3	0.2	0.3	

(b) The possible values of Z are 5 through 9;
$(p_Z(5), p_Z(6), p_Z(7), p_Z(8), p_Z(9)) = (0.3,\ 0.1,\ 0.4,\ 0.1,\ 0.1)$.

(c) No. For example, $p_{X,Y}(0,4) = 0 \neq 0.08 = p_X(0)p_Y(4)$.

(d) Normalizing the column for $u = 3$ yields that $p_{Y|X}(v|3)$ is equal to $\frac{2}{3}$ for $v = 4$, $\frac{1}{3}$ for $v = 6$, and zero for other values of v. Therefore, $E[Y|X = 3] = 4 \cdot \frac{2}{3} + 6 \cdot \frac{1}{3} = \frac{14}{3} = 4.666\ldots$.

2. [**A joint distribution**]

(a) Clearly $f_X(u) = 0$ for $u < 0$. For $u \geq 0$,

$$f_X(u) = \int_0^\infty ve^{-(1+u)v}dv = \frac{1}{(1+u)^2}\int_0^\infty (1+u)^2 ve^{-(1+u)v}dv = \frac{1}{(1+u)^2},$$

where we used the fact that the gamma density with parameters $r = 2$ and $\lambda = 1 + u$ integrates to one.
Clearly $f_Y(v) = 0$ for $v < 0$. For $v > 0$,

$$f_V(v) = \int_0^\infty ve^{-(1+u)v}du = ve^{-v}\int_0^\infty e^{-uv}du = ve^{-v}\frac{1}{v} = e^{-v}.$$

That is, V has the exponential distribution with parameter one.

(b) Since the support of f_X is $\mathbb{R}_+$, the conditional pdf $f_{Y|X}(v|u)$ is well-defined only for $u \geq 0$. For such u,

$$f_{Y|X}(v|u) = \begin{cases} \frac{ve^{-(1+u)v}}{\frac{1}{(1+u)^2}} = (1+u)^2 ve^{-(1+u)v} & v \geq 0 \\ 0 & v < 0. \end{cases}$$

That is, the conditional distribution of Y given $X = u$ is the gamma distribution with parameters $r = 2$ and $\lambda = 1 + u$.
Since the support of f_Y is $\mathbb{R}_+$, the conditional pdf $f_{X|Y}(u|v)$ is well-defined only for $v \geq 0$. For such v,

$$f_{X|Y}(u|v) = \begin{cases} \frac{ve^{-(1+u)v}}{e^{-v}} = ve^{-uv} & u \geq 0 \\ 0 & u < 0. \end{cases}$$

That is, the conditional distribution of X given $Y = v$ is the exponential distribution with parameter v.

(c) The joint CDF $F_{X,Y}(u_o, v_o)$ is zero if either $u_o < 0$ or $v_o < 0$. For $u_o \geq 0$ and $v_o \geq 0$,

$$\begin{aligned} F_{X,Y}(u_0, v_0) &= \int_0^{v_o}\int_0^{u_o} ve^{-(1+u)v}dudv \\ &= \int_0^{v_o} e^{-v}\left\{\int_0^{u_o} ve^{-uv}du\right\}dv \\ &= \int_0^{v_o} e^{-v}(1-e^{-u_o v})dv \\ &= \int_0^{v_o} (e^{-v} - e^{-v(1+u_o)})dv \\ &= \frac{1}{1+u_o}\left\{u_o + e^{-v_o(1+u_o)}\right\} - e^{-v_o} \end{aligned}$$

(d) No. For example, the answer to part (b) shows that $f_{Y|X}(v|u)$ is well defined for $u > 0$ and it is not a function of v alone.

3. **[Recognizing independence]**

(a) Yes. $f_X(u) = f_Y(u) = \begin{cases} \frac{2}{\sqrt{\pi}}e^{-u^2} & u \geq 0 \\ 0 & \text{else.} \end{cases}$ Note: X and Y each have the same distribution as the absolute value of a $N(0, 0.5)$ random variable.

(b) Yes. $f_X(u) = \begin{cases} -\ln(u) & u \geq 0 \\ 0 & \text{else} \end{cases}$ and $f_Y(v) = \begin{cases} \frac{v^2}{21} & 1 \leq v \leq 4 \\ 0 & \text{else.} \end{cases}$

(c) No. The support set of $f_{X,Y}$ is the unit disk, $\{(u,v) : u^2 + v^2 \leq 1\}$, which is not a product set. For example, $(0, 0.8)$ and $(0.8, 0)$ are in the support of $f_{X,Y}$ but $(0.8, 0.8)$ is not.

4. **[Joint pmfs]**

(a) The marginal pmfs $p_{\mathbb{X}}(u)$ and $p_{\mathbb{Y}}(v)$ are column and row sums as shown in the table below.

$u \rightarrow$ / $v \downarrow$	0	1	3	5	Row sum
4	0	1/12	1/6	1/12	1/3
3	1/6	1/12	0	1/12	1/3
-1	1/12	1/6	1/12	0	1/3
Column sum	1/4	1/3	1/4	1/6	1

(b) The eyeball test tells us that $\mathbb{X}$ and $\mathbb{Y}$ are *dependent* random variables. Less optically, $p_{\mathbb{X},\mathbb{Y}}(0,4) = 0 \neq p_{\mathbb{X}}(0)p_{\mathbb{Y}}(4) = \frac{1}{4} \times \frac{1}{3} = \frac{1}{12}$ and so $\mathbb{X}$ and $\mathbb{Y}$ are not independent random variables.

(c) $P\{\mathbb{X} \leq \mathbb{Y}\} = p_{\mathbb{X},\mathbb{Y}}(0,3) + p_{\mathbb{X},\mathbb{Y}}(0,4) + p_{\mathbb{X},\mathbb{Y}}(1,3) + p_{\mathbb{X},\mathbb{Y}}(1,4) + p_{\mathbb{X},\mathbb{Y}}(3,4) = \frac{1}{2}$.

$P\{\mathbb{X} + \mathbb{Y} \leq 4\} = p_{\mathbb{X}}(0) + p_{\mathbb{X},\mathbb{Y}}(1,3) + p_{\mathbb{X},\mathbb{Y}}(1,-1) + p_{\mathbb{X},\mathbb{Y}}(3,-1) + p_{\mathbb{X},\mathbb{Y}}(5,-1) = \frac{7}{12}$.

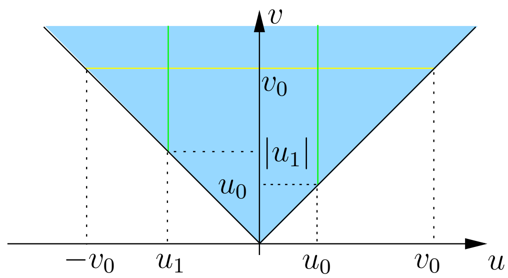
(d) $p_{\mathbb{X}|\mathbb{Y}}(u|3) = \dfrac{p_{\mathbb{X},\mathbb{Y}}(u,3)}{p_{\mathbb{Y}}(3)} = 1/2, 1/4, 1/4$ respectively for $u = 0, 1, 5$.

$\mathsf{E}[\mathbb{X} \mid \mathbb{Y} = 3] = 0 \times \frac{1}{2} + 1 \times \frac{1}{4} + 5 \times \frac{1}{4} = \frac{3}{2}$.

$\mathsf{var}(\mathbb{X} \mid \mathbb{Y} = 3) = \mathsf{E}[\mathbb{X}^2 \mid \mathbb{Y} = 3] - (\mathsf{E}[\mathbb{X} \mid \mathbb{Y} = 3])^2 = 1 \times \frac{1}{4} + 5^2 \times \frac{1}{4} - \left(\frac{3}{2}\right)^2 = \frac{26}{4} - \frac{9}{4} = \frac{17}{4}$.

5. []

The pdf is nonzero over the shaded region in the figure shown below.



(a) From the figure, we get that for any $v_0 > 0$,

$$f_{\mathbb{Y}}(v_0) = \int_{u=-v_0}^{u=+v_0} c(v^2 - u^2)\exp(-v)\,du = c\left[v^2u - \frac{u^3}{3}\right]\exp(-v)\Big|_{u=-v_0}^{u=+v_0} = c\left(\frac{4}{3}\right)v_0^3\exp(-v_0)$$

which is of the form of a *gamma* pdf with parameters $(4, 1)$.
Thus, $4c/3 = 1/\Gamma(4) = 1/3! \Rightarrow c = 1/8$.

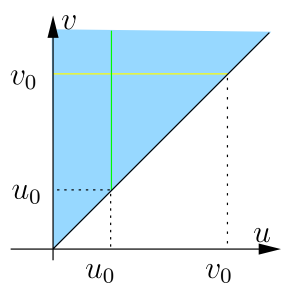
(b) For any $u_0 > 0$, $f_{\mathbb{X}}(u_0) = \int_{v=u_0}^{\infty} \frac{1}{8}(v^2 - u^2)\exp(-v)\,dv = \frac{1}{4}(1 + u_0)\exp(-u_0)$, while for $u_1 < 0$, the limits are $v = |u_1|$ and ∞ giving $f_{\mathbb{X}}(u_1) = \frac{1}{4}(1 + |u_1|)\exp(-|u_1|)$. Consequently, $f_{\mathbb{X}}(u) = \frac{1}{4}(1 + |u|)\exp(-|u|), -\infty < u < \infty$.

The marginal density/pdf of $\mathbb{Y}$ was found in part (a) as $f_{\mathbb{Y}}(v) = \begin{cases} \frac{v^3}{3!}\exp(-v), & v > 0, \\ 0, & \text{otherwise.} \end{cases}$

(c) The pdf of $\mathbb{X}$ is an even function of u and $\int_0^{\infty} u \cdot f_{\mathbb{X}}(u)\,du$ is finite. Hence, $\mathsf{E}[\mathbb{X}] = 0$.

6. **[Drill problem]**

(a) The joint pdf is nonzero on the shaded region shown in the figure below.



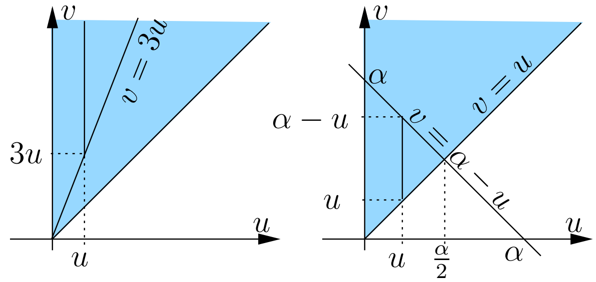
For any $u_0 > 0$, $f_{\mathbb{X}}(u_0) = \int_{v=u_0}^{\infty} 2\exp(-u_0 - v)\,dv = 2\exp(-2u_0)$.

For any $v_0 > 0$, $f_{\mathbb{Y}}(v_0) = \int_{u=0}^{v_0} 2\exp(-u - v_0)\,du = 2\exp(-v_0) - 2\exp(-2v_0)$. Consequently,

$$f_{\mathbb{X}}(u) = \begin{cases} 2\exp(-2u), & u > 0, \\ 0, & \text{elsewhere,} \end{cases} \quad \text{and} \quad f_{\mathbb{Y}}(v) = \begin{cases} 2\exp(-v) - 2\exp(-2v), & v > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

(b) No, the eyeball test says that the random variables are dependent. Less optically, $f_{\mathbb{X},\mathbb{Y}}(u, v) = 0 \neq f_{\mathbb{X}}(u)f_{\mathbb{Y}}(v)$ for any u and v such that $0 < v < u$.

(d) $P\{\mathbb{Y} > 3\mathbb{X}\} = \int_{u=0}^{\infty}\int_{v=3u}^{\infty} 2\exp(-u - v)\,dv\,du = \int_{u=0}^{\infty} 2\exp(-4u)\,du = \frac{1}{2}$.
See the left-hand figure below.



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