ECE 313: Problem Set 9: Solutions
Functions of a Random Variable, Maximum Likelihood Estimation, Hypothesis Testing

1. []

(a) $\mathcal{I}$ can take on values in the range $(-I_0, \infty)$.

(b) $F_{\mathcal{I}}(v) = 0$ for $v < -I_0$. For any $v > -I_0$,
$F_{\mathcal{I}}(v) = P\{\mathcal{I} \le v\} = P\{I_0(\exp(\mathcal{V}) - 1) \le v\} = P\{\mathcal{V} \le \ln(1 + v/I_0)\} = F_{\mathcal{V}}(\ln(1 + v/I_0))$.

(c) For $v > -I_0$,
$f_{\mathcal{I}}(v) = f_{\mathcal{V}}(\ln(1+v/I_0))\dfrac{1}{1+v/I_0} \times \dfrac{1}{I_0} = \dfrac{f_{\mathcal{V}}(\ln(1+v/I_0))}{v+I_0} = \begin{cases} \frac{I_0/2}{(v+I_0)^2}, & v \ge 0, \\ \frac{1}{2I_0}, & -I_0 < v < 0, \end{cases}$.
Note that the pdf has constant value $1/(2I_0)$ from $v = -I_0$ to $v = 0$.

2. []

(a) $\mathbb{Y}$ can take on only two values in the set $\{-\alpha, \alpha\}$. Further $P\{\mathbb{Y} = -\alpha\} = P\{\mathbb{X} \le 0\} = 0.5$. It follows that $P\{\mathbb{Y} = \alpha\} = P\{\mathbb{X} > 0\} = 0.5$.

(b) Using LOTUS

$$\begin{aligned}
\mathsf{E}[\mathbb{Z}] &= \int_{-\infty}^{0} (x+\alpha)^2 f_{\mathbb{X}}(x)\,dx \quad + \quad \int_{0}^{\infty} (x-\alpha)^2 f_{\mathbb{X}}(x)\,dx \\
&= \int_{-\infty}^{\infty} (x^2+\alpha^2) f_{\mathbb{X}}(x)\,dx \quad - \quad 2\alpha\int_0^\infty x\, f_{\mathbb{X}}(x)\,dx \quad + \quad 2\alpha \int_{-\infty}^{0} x\ f_{\mathbb{X}}(x)\,dx \\
&= (1+\alpha^2) - 4\alpha\int_0^\infty x\, f_{\mathbb{X}}(x)\,dx \\
&= (1+\alpha^2) - \frac{4\alpha}{\sqrt{2\pi}}\int_0^\infty x\, e^{-\frac{x^2}{2}}\,dx \\
&= (1+\alpha^2) - \frac{4\alpha}{\sqrt{2\pi}}.
\end{aligned}$$

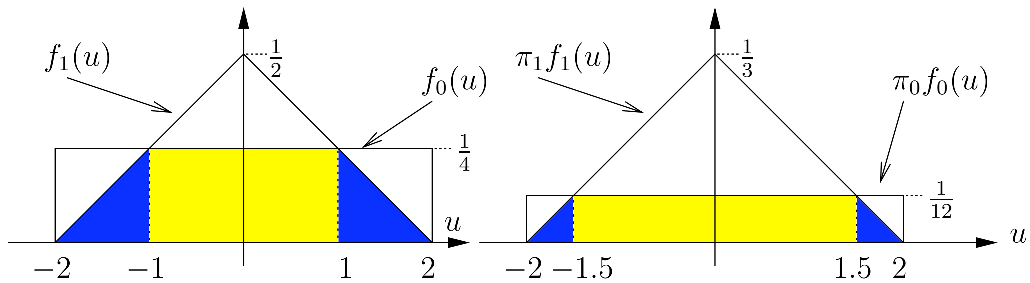
The choice of α that minimizes $\mathsf{E}[\mathbb{Z}]$ is $\frac{2}{\sqrt{2\pi}}$.

(c) $\mathbb{W}$ takes values in $\{\pm 3, \pm 2, \pm 1, 0\}$. The probabilities are

$$\begin{aligned}
P\{\mathbb{W} = 0\} &= P\{-0.5 \le \mathbb{X} < 0.5\} = 1 - 2Q(0.5) \\
P\{\mathbb{W} = 1\} &= P\{0.5 \le \mathbb{X} < 1.5\} = Q(0.5) - Q(1.5) \\
P\{\mathbb{W} = -1\} &= P\{-1.5 \le \mathbb{X} < -0.5\} = Q(0.5) - Q(1.5) \\
P\{\mathbb{W} = 2\} &= P\{1.5 \le \mathbb{X} < 2.5\} = Q(1.5) - Q(2.5) \\
P\{\mathbb{W} = 2\} &= P\{-2.5 \le \mathbb{X} < -1.5\} = Q(1.5) - Q(2.5) \\
P\{\mathbb{W} = 3\} &= P\{2.5 \ge \mathbb{X}\} = Q(2.5) \\
P\{\mathbb{W} = -3\} &= P\{-2.5 > \mathbb{X}\} = Q(2.5).
\end{aligned}$$

3. []

(a) The easiest way to solve this problem is to sketch the two pdfs as shown in the left-hand figure below.



It is obvious that the maximum-likelihood decision is in favor of H_1 if $|\mathbb{X}| < 1$, and hence $x = 0,\ \eta = 1$. By inspection, we get that $P_{\text{FA}} = 2 \times \frac{1}{4} = \frac{1}{2}$ while $P_{\text{MD}} = 2 \times \left(\frac{1}{2} \times 1 \times \frac{1}{4}\right) = \frac{1}{4}$.

The graphically-challenged can proceed as follows.
For $-2 < u < 2$, the likelihood ratio is $\Lambda(u) = \dfrac{f_1(u)}{f_0(u)} = \dfrac{0.25(2-|u|)}{0.25} = 2 - |u|$.
When $\mathbb{X} = u$ is the observation, the *maximum-likelihood* decision rule decides in favor of H_1 if $\Lambda(u) > 1$. Hence $\Gamma_1 = \{u\ :\ |u| < 1\}$ and $\Gamma_0 = \{u\ :\ 1 < |u| < 2\}$, that is, the ML decision rule is that if $|\mathbb{X}| > 1$, the decision is that H_0 is the true hypothesis. Thus, we have $x = 0$, and $\eta = 1$.
$P_{\text{FA}} = \displaystyle\int_{\Gamma_1} f_0(u)\,du = \int_{-1}^{1} \frac{1}{4} du = \frac{1}{2}$.
$P_{\text{MD}} = \displaystyle\int_{\Gamma_0} f_1(u)\,du = 2\int_{1}^{2} \frac{1}{4}(2-u)\,du = \frac{1}{2}\left(2u - \frac{u^2}{2}\right)\Bigg|_1^2 = \frac{1}{4}$.

(b) The probability of error of the ML decision rule is

$$P(E) = \pi_0 P_{\text{FA}} + \pi_1 P_{\text{MD}} = \frac{1}{3} \times \frac{1}{2} + \frac{2}{3} \times \frac{1}{4} = \frac{1}{3}.$$

(c) Sketching $\pi_0 f_0(u)$ and $\pi_1 f_1(u)$ as in the right-hand figure above, we easily see that the MAP decision is in favor of H_1 if $|\mathbb{X}| < 1.5$, and hence $x = 0,\ \xi = 1.5$. By inspection, we get that $\pi_0 P_{\text{FA}} = 3 \times \frac{1}{12} = \frac{1}{4} = \frac{1}{3} \times \frac{3}{4}$ while $\pi_1 P_{\text{MD}} = 2 \times \left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{12}\right) = \frac{1}{24} = \frac{2}{3} \times \frac{1}{16}$, that is, $P_{\text{FA}} = \frac{3}{4}$, and $P_{\text{MD}} = \frac{1}{16}$. $P(E) = \pi_0 P_{\text{FA}} + \pi_1 P_{\text{MD}} = \frac{1}{4} + \frac{1}{24} = \frac{7}{24} < \frac{1}{3}$, where $\frac{1}{3}$ is the error probability of the ML rule (with the same *a priori* probabilities) that we found in part (c).

Without using any graphical aids, we have that when $\mathbb{X} = u$ is the observation, the MAP decision rule decides in favor of H_1 if $\Lambda(u) = 2 - |u| > \pi_0/\pi_1 = 1/2$. Hence, $\Gamma_1 = \left\{u\ :\ |u| < \frac{3}{2}\right\}$ and $\Gamma_0 = \left\{u\ :\ \frac{3}{2} < |u| < 2\right\}$ for the MAP decision rule. Once again, $x = 0$ while $\xi = \frac{3}{2}$. We get
$P_{\text{FA}} = \displaystyle\int_{\Gamma_1} f_0(u)\,du = \int_{-\frac{3}{2}}^{\frac{3}{2}} \frac{1}{4} du = \frac{3}{4}$.
$P_{\text{MD}} = \displaystyle\int_{\Gamma_0} f_1(u)\,du = 2\int_{\frac{3}{2}}^{2} \frac{1}{4}(2-u)\,du = \frac{1}{2}\left(2u - \frac{u^2}{2}\right)\Bigg|_{\frac{3}{2}}^2 = \frac{1}{16}$.
Hence, $P(E) = \pi_0 \cdot P_{\text{FA}} + \pi_1 \cdot P_{\text{MD}} = \dfrac{1}{3} \times \dfrac{3}{4} + \dfrac{2}{3} \times \dfrac{1}{16} = \dfrac{1}{4} + \dfrac{1}{24} = \dfrac{7}{24} < \dfrac{1}{3}$.

4. []

(a) The average radius of the sphere is $\int_0^1 u2u\,du = \frac{2}{3}$.

(b) The average volume is $\int_0^1 \frac{4\pi u^3}{3} \cdot 2u\,du = \frac{8\pi}{15}$.

(c) The average surface area is $\int_0^1 4\pi u^2 \cdot 2u\,du = 2\pi$.

(d) The average sphere has volume $\frac{32\pi}{91}$ which is larger than the average volume. The average sphere has surface area $\frac{16\pi}{9}$ which is larger than the average surface area.

5. []

Suppose the support of $\mathbb{X}$ is $[a, b]$. Then $a+b = 2$ and $(b-a)^2 = 36$. This allows us to calculate $a = -2$ and $b = 4$. The pdf $f_{\mathbb{Y}}(y)$is naturally divided into two regions: $0 \leq y \leq 2$ and $4 \geq y > 2$:

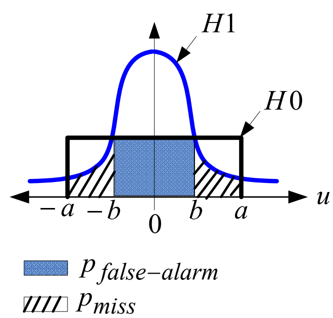
$$f_{\mathbb{Y}}(y) = \begin{cases} \frac{1}{3} & 0 \leq y \leq 2 \\ \frac{1}{6} & 2 < y \leq 4 \\ 0 & y > 4. \end{cases}$$

Formal derivation is as follows: For $y < 0$, the CDF $F_{\mathbb{Y}}(y) = 0$. Further for $y > 4$, the CDF $F_{\mathbb{Y}}(y) = 1$.
For any $0 \leq y \leq 2$, the CDF $F_{\mathbb{Y}}(y) = P\{\mathbb{Y} \leq y\} = P\{-y \leq \mathbb{X} \leq y\} = \frac{y}{3}$.
For any $2 \leq y \leq 4$, the CDF $F_{\mathbb{Y}}(y) = P\{\mathbb{Y} \leq y\} = P\{\mathbb{X} \leq y\} = \frac{y+2}{6}$.
Now $f_{\mathbb{Y}}(y) = \frac{d}{dy} F_{\mathbb{Y}}(y)$ follows directly.

6. []

(a) From the figure, the pdf for H_1 is smaller than the pdf for H_0 precisely when $b < |u| < a$. Thus, the ML rule is given by:

$$\widehat{H} = \begin{cases} H_0 & b < |X| < a \\ H_1 & \text{otherwise} \end{cases}$$



(b)

(c) These probabilities are calculated as follows:

$$\begin{aligned} p_{false\ alarm} &= \frac{2b}{2a} = \frac{b}{a} \\ p_{miss} &= 2(\Phi(a) - \Phi(b)) = 2(Q(b) - Q(a)) \end{aligned}$$

(d) Given π_1, one obtains

$$\begin{aligned} \pi_0 &= 1 - \pi_1 = \frac{\sqrt{3}}{3 + \sqrt{2\pi}} \\ \frac{\pi_0}{\pi_1} &= \frac{3}{\sqrt{2\pi}} \end{aligned}$$

The LRT gives us

$$\begin{aligned} \frac{3}{\sqrt{2\pi}} e^{\frac{-u^2}{2}} &= \frac{3}{\sqrt{2\pi}} \\ e^{\frac{-u^2}{2}} = 1 \end{aligned}$$

Thus, the MAP rule is given by

$$\hat{H} = \begin{cases} H1 & u > 0 \\ H0 & otherwise \end{cases}$$