ECE 313: Problem Set 8: Solutions
Poisson process, Linear scaling, Gaussian distribution

1. **[Poisson process]**

   Model the access times as a Poisson process indexed by time measured in seconds, with rate $\lambda = \frac{15}{60} = 1/4$. The probability to be found is

$$\begin{aligned} P\{N_{10} = 3, N_{60} - N_{45} = 2\} &= P\{N_{10} = 3\}P\{N_{60} - N_{45} = 2\} \\ &= \left(\frac{(10/4)^3 e^{-10/4}}{3!}\right)\left(\frac{(15/4)^2 e^{-15/4}}{2!}\right) \\ &= 0.0353 \end{aligned}$$

2. **[Poisson process probabilities]**

   (a) The numbers of arrivals in the disjoint intervals are independent Poisson random variables with mean λ. Thus, the probability is $(\lambda e^{-\lambda})^3 = \lambda^3 e^{-3\lambda}$.

   (b) We wish to determine the probability of the event $A = \{\mathbb{N}(0,2] = 2, \mathbb{N}(1,3] = 2\}$. Note that A is the union of the mutually exclusive events B_{020}, B_{111}, and B_{202} where

$$B_{ijk} = \{\mathbb{N}(0,1] = i, \mathbb{N}(1,2] = j, \mathbb{N}(2,3] = k\}.$$

   So, since the events $\{\mathbb{N}(0,1] = i\}$, $\{\mathbb{N}(1,2] = j\}$ and $\{\mathbb{N}(2,3] = k\}$ are independent events,

$$\begin{aligned} P(A) &= P(B_{020}) + P(B_{111}) + P(B_{202}) \\ &= e^{-\lambda}\cdot\left(\frac{\lambda^2}{2!}e^{-\lambda}\right)\cdot e^{-\lambda} + \left(\lambda e^{-\lambda}\right)^3 + \left(\frac{\lambda^2}{2!}e^{-\lambda}\right)\cdot e^{-\lambda}\cdot\left(\frac{\lambda^2}{2!}e^{-\lambda}\right) = \left(\frac{\lambda^2}{2} + \lambda^3 + \frac{\lambda^4}{4}\right)e^{-3\lambda} \end{aligned}$$

   (c) Here, we are asked for the *conditional* probability $P(\{\mathbb{N}(1,2) = 2\} \mid A) = \dfrac{P(\{\mathbb{N}(1,2) = 2\} \cap A)}{P(A)}$. But, $\{\mathbb{N}(1,2) = 2\} \cap A = B_{020}$ and so

$$P(\{\mathbb{N}(1,2) = 2\} \mid A) = \frac{P(B_{020})}{P(A)} = \frac{\frac{\lambda^2}{2}}{\frac{\lambda^2}{2} + \lambda^3 + \frac{\lambda^4}{4}} = \frac{2}{2 + 4\lambda + \lambda^2}.$$

3. **[Gaussian Distribution]**

   (a) As X is a continuous-type random variable, $P\{X = c\} = 0$ for any value of c including $c = 0$.

   (b)

$$\begin{aligned} P\{|X+4| \geq 2\} &= P\{X \leq -6 \cup X \geq -2\} \\ &= P\{X \leq -6\} + P\{X \geq -2\} = 2P\{X \geq -2\} \\ &= 2P\{\frac{X+4}{3} \geq 2/3\} = 2Q(2/3) = 2 \times 0.2514 = 0.5028 \end{aligned}$$

   (c)

$$\begin{aligned} P\{0 < X < 2\} &= P\{X > 0\} - P\{X \geq 2\} \\ &= P\{\frac{X+4}{3} > 4/3\} - P\{\frac{X+4}{3} \geq 2\} \\ &= Q(4/3) - Q(2) = 0.0918 - 0.0228 = 0.069 \end{aligned}$$

(d)

$$\begin{aligned} P\{X^2 < 9\} &= P\{-3 < X < 3\} \\ &= P\{\frac{1}{3} < \frac{X+4}{3} < 7/3\} \\ &= \Phi(7/3) - \Phi(1/3) \approx \Phi(2.33) - \Phi(0.33) = 0.9901 - 0.6293 = 0.3608 \end{aligned}$$

4. **[Enhancing the Yield of Cell Phones]**

(a) As $V_A = 1 + IR$ and $I = 10^{-3}$,

$$\begin{aligned} E[V_A] &= E[1+IR] = 1 + IE[R] = 1 + 10^{-3} \times 10^3 = 2V \\ Var[V_A] &= Var[1+IR] = I^2 Var[R] = 10^{-6} \times 10^4 = 10^{-2}V^2 \end{aligned}$$

(b) As R has a $\mathcal{N}(1k\Omega, 10^4\Omega^2)$ distribution, V_A has a $\mathcal{N}(2V, 10^{-2}V^2)$ distribution as it is a linearly scaled version of R. The yield Y is simply the probability $P\{1.95V \leq V_A \leq 2.05V\}$. As the Gaussian distribution is symmetric about its mean, $Y = 1 - 2P\{V_A > 2.05V\}$.

$$P\{V_A > 2.05V\} = P\{\frac{V_A - 2}{0.1} > 0.5\} = Q(0.5) = 0.3085$$

Thus, the fraction of cell phones that will work is $1 - 0.617 = 0.383$, i.e., the yield is $Y = 38.3\%$.

(c) To obtain $Y = 90\%$ requires that $P\{V_A > 2.05V\} = (1 - 0.9)/2 = 0.05$. From Table 5.2, we find that $Q(1.64) = 0.05$. Thus,

$$\begin{aligned} P\{V_A > 2.05V\} &= 0.05 = P\{\frac{V_A - 2}{\sigma_{V_A}} > 1.64\} \\ \sigma_{V_A} &< \frac{V_A - 2}{1.64} = \frac{0.05}{1.64} = 0.03 = 3 \times 10^{-2} \end{aligned}$$

Thus, $\sigma^2_{V_A} < 9 \times 10^{-4}V^2$. With $\sigma^2_{V_A} = 10^{-6}\sigma^2_R$, we get $\sigma^2_R = 900\Omega^2$. Thus, the variance of this precision resistor needs to be at most $900\Omega^2$ in order to achieve a yield greater than 90%. That is, the required standard deviation is 30Ω, or 3% of the desired 1000Ω resistance, whereas the original transistors had a standard deviation of 100Ω, or 10% of the desired resistance. Hopefully, the cost of the higher precision resistors is low enough to keep the cell phone competitive in the market place.

5. **[DeMoivre-Laplace approximation to central term of binomial distribution]**

(a) Note that $\mathsf{var}(\mathbb{X}) = np(1-p) = n/4$.

$$\begin{aligned} P\left\{\mathbb{X} = \frac{n}{2}\right\} &= P\left\{\frac{n}{2} - \frac{1}{2} \leq \mathbb{X} \leq \frac{n}{2} + \frac{1}{2}\right\} \\ &= P\left\{-\frac{1}{2} \leq \mathbb{X} - \frac{n}{2} \leq \frac{1}{2}\right\} \\ &= P\left\{-\frac{1}{\sqrt{n}} \leq \frac{\mathbb{X} - n/2}{\sqrt{n/4}} \leq \frac{1}{\sqrt{n}}\right\} \\ &\approx \Phi\left(\frac{1}{\sqrt{n}}\right) - \Phi\left(-\frac{1}{\sqrt{n}}\right) = 2\left(\Phi\left(\frac{1}{\sqrt{n}}\right) - 0.5\right) \end{aligned}$$

(b) Since $\Phi(0) = 0.5$ and the derivative of Φ is the standard normal density, so that $\Phi'(0) = \frac{1}{\sqrt{2\pi}}$, we get

$$\lim_{n\to\infty} \sqrt{n} \times 2\left(\Phi\left(\frac{1}{\sqrt{n}}\right) - 0.5\right) = \lim_{n\to\infty} 2\left(\frac{\Phi\left(\frac{1}{\sqrt{n}}\right) - \Phi(0)}{\frac{1}{\sqrt{n}}}\right) = 2\Phi'(0) = \sqrt{\frac{2}{\pi}} \approx 0.7979$$

In other words, $P\left\{\mathbb{X} = \frac{n}{2}\right\} \approx \sqrt{\frac{2}{\pi n}}$ for large values of n.

(c) For $n = 30$, we have

exact value	$P\{\mathbb{X} = 15\} = \binom{30}{15} 2^{-30}$	0.144464
approximation from (a)	$\Phi(0.18) - \Phi(-0.18)$	0.144868
approximation from (b)	$\sqrt{\frac{2}{30\pi}} = \frac{1}{\sqrt{15\pi}}$	0.145673