ECE 313: Problem Set 7: Solutions
CDFs, continuous-type random variables, uniform and exponential distributions

1. **[CDFs]** The functions shown in plots (b) and (d) are valid CDFs and the other four are not. The function in (a) does not converge to zero at $-\infty$ nor to one at $+\infty$. Plot (c) doesn't even show a function–for some values of u there are two values of $F(u)$ shown. The function in (e) is not right continuous. The function in (f) is not nondecreasing.

2. **[CDFs]**

   (a) $P\{X = 0\} = F_X(0) - F_X(0^-) = 0$.

   (b) $P\{X = 10\} = F_X(10) - F_X(10^-) = 1 - 0.75 = 0.25$.

   (c) $P\{X < 10\} = F_X(10^-) = 0.75$.

   (d) $P\{X \geq -6\} = 1 - P\{X < -6\} = 1 - F_X(-6^-) = 1 - 0.25 = 0.75$.

   (e) $P\{|X| < -10\} = 0$.

   (f) $P\{X^2 \leq 16\} = P\{-4 \leq X \leq 4\} = F_X(4) - F_X(-4^-) = 0.7 - 0.3 = 0.4$.

3. **[Continuous-type random variables]**

   (a) Check two properties:
   $f(x) \geq 0 \ \surd$
   $\int_{-\infty}^{\infty} f(x)dx = \int_1^{\infty} x^{-1}dx = \ln(\infty) \neq 1 \ \times$
   One of the properties is not satisfied, therefore $f(x)$ is not a valid pdf. Also, there is no constant c such that $cf(x)$ is a valid pdf.

   (b) Check two properties:
   $f(x) \geq 0 \ \times$ because $\ln(x) < 0$ for $0 < x < 1$.
   One of the properties is not satisfied, therefore $f(x)$ is not a valid pdf. However, if we choose $c = -1$ then
   $cf(x) \geq 0 \ \surd$
   $\int_{-\infty}^{\infty} cf(x)dx = \int_0^1 (-\ln(x))dx = -(x(\ln(x) - 1))|_0^1 = 1 \ \surd$
   Both of the properties are satisfied if $c = -1$, therefore $-f(x)$ is a valid pdf.

   (c) Check two properties:
   $f(x) \geq 0 \ \times$ because $(x - 1) < 0$ for $x \in (0, 1)$
   One of the properties is not satisfied, therefore $f(x)$ is not a valid pdf. Also, there is no constant c such that $cf(x)$ is a valid pdf.

   (d) Check two properties:
   $f(x) \geq 0 \ \surd$
   $\int_{-\infty}^{\infty} f(x)dx = \int_{-1}^{0}(-x)dx + \int_0^1 xdx = 1 \ \surd$
   Both properties are satisfied, therefore $f(x)$ is a valid pdf.

   (e) Check two properties:
   $f(x) \geq 0 \ \surd$
   $\int_{-\infty}^{\infty} f(x)dx = \int_0^{\infty} e^{-2x}dx = \frac{1}{2} \ \times$
   One of the properties is not satisfied, therefore $f(x)$ is not a valid pdf. However, if we choose $c = 2$ then
   $cf(x) \geq 0 \ \surd$
   $\int_{-\infty}^{\infty} cf(x)dx = \int_0^{\infty} 2e^{-2x}dx = 1 \ \surd$
   Both of the properties are satisfied if $c = 2$, therefore $2f(x)$ is a valid pdf.

(f) Check two properties:
$f(x) \geq 0$ ✓
$\int_{-\infty}^{\infty} f(x)dx = \int_0^\infty x^2 e^{-x} dx = -e^{-x}(x^2+2x+2)|_0^\infty = 2$ ×
One of the properties is not satisfied, therefore $f(x)$ is not a valid pdf. However, if we choose $c = 1/2$ then
$cf(x) \geq 0$ ✓
$\int_{-\infty}^{\infty} cf(x)dx = \int_0^\infty \frac{1}{2}x^2 e^{-x} dx = -\frac{1}{2}e^{-x}(x^2+2x+2)|_0^\infty = 1$ ✓
Both of the properties are satisfied if $c = 1/2$, therefore $f(x)/2$ is a valid pdf.

4. **[Continuous-type random variables]**

(a) There are three unknowns: a, b, c, and we have three equations that have to be satisfied:

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} f(u)du = \int_2^3 (a + bu + cu^2)du = a + \frac{5}{2}b + \frac{19}{3}c \\
\frac{5}{2} &= \int_{-\infty}^{\infty} uf(u)du = \int_2^3 u(a + bu + cu^2)du = \frac{5}{2}a + \frac{19}{3}b + \frac{65}{4}c \\
\frac{1}{12} + \left(\frac{5}{2}\right)^2 = \frac{19}{3} &= \int_{-\infty}^{\infty} u^2(u)du = \int_2^3 u^2(a + bu + cu^2) = \frac{19}{3}a + \frac{65}{4}b + \frac{211}{5}c
\end{aligned}$$

One can solve the above system of equations in various ways. One is to notice that the constant on the left-hand side of all three equations is the same as the coefficient of a. Hence, the solution is $a = 1$ and $b = c = 0$.

(b) The pdf of X is zero outside $[2, 3] \subset [1, 4)$, therefore $P\{1 \leq X < 4\} = 1$.

(c) Part (a) showed that X is a uniform random variable on $[2, 3]$ with mean $5/2$, therefore $P\{X > 5/2\} = 1/2$.

5. **[Uniform and exponential distributions]**

(a) X is a continuous-type random variable, therefore $P\{X = 5\} = 0$.

(b) The CDF of Y is $F_Y(y) = 1 - e^{-2y}$ for $y \geq 0$ and one otherwise. By the memoryless property of the exponential distribution $P\{Y > 3|Y > 1\} = P\{Y > 2\} = 1 - F_Y(2) = e^{-4}$.

(c) The CDF of X is $F(x) = \frac{x-1}{9}$ for $x \in [1, 10]$, it is 1 for $x > 10$ and zero otherwise. Therefore, $P\{1.0 \leq X < 2.0\} = F(2.0^-) - F(1.0^-) = \frac{2-1}{9} - \frac{1-1}{9} = \frac{1}{9}$

(d)

$$\begin{aligned}
&P\{Y \in [0.2, 0.3) \cup [1.2, 1.3) \cup [2.2, 2.3) \cup \ldots\} = P\{0.2 \leq Y < 0.3\} + P\{1.2 \leq Y < 1.3\} + \ldots \\
&= \sum_{k=0}^{\infty} \left[F\left((k + 0.3)^-\right) - F\left((k + 0.2)^-\right)\right] = \sum_{k=0}^{\infty} \left(-e^{-2(k+0.3)} + e^{-2(k+0.2)}\right) \\
&= \left(-e^{-0.6} + e^{-0.4}\right) \sum_{k=0}^{\infty} e^{-2k} = \left(-e^{-0.6} + e^{-0.4}\right) \frac{1}{1 - e^{-2}} \approx 0.1405
\end{aligned}$$

(e) Notice that $(0.4)^2 < 1$ and that $(3.3)^2 > 10$, therefore,

$$\begin{aligned}
&P\left\{\left\{1.3 \leq \sqrt{X} < 1.4\right\} \cup \left\{2.3 \leq \sqrt{X} < 2.4\right\}\right\} \\
&= P\left\{\left\{1.69 \leq \sqrt{X} < 1.96\right\} \cup \left\{5.29 \leq \sqrt{X} < 5.76\right\}\right\} \\
&= \left(F(1.96^-) - F(1.69^-)\right) + \left(F(5.76^-) - F(5.29^-)\right) \\
&= \frac{1.96 - 1.69}{9} + \frac{5.76 - 5.29}{9} = \frac{0.74}{9} = \frac{37}{450} \approx 0.0822
\end{aligned}$$