ECE 313: Problem Set 6: Solutions
Decision Making Under Uncertainty, Reliability

1. **[Detection problem with a geometric model]**

(a) If $\mathbb{X} = n$, the likelihood ratio has value

$$\Lambda(n) = \frac{p_1(1-p_1)^{n-1}}{p_0(1-p_0)^{n-1}} = \frac{p_1}{p_0}\left(\frac{1-p_1}{1-p_0}\right)^{n-1} > 1 \text{ if } (n-1)\ln\left(\frac{1-p_1}{1-p_0}\right) > \ln\left(\frac{p_0}{p_1}\right)$$

Since $p_1 < p_0$, we have that $1-p_1 > 1-p_0$ and $\ln((1-p_1)/(1-p_0)) > 0$. Therefore, the maximum likelihood decision rule is

$$\text{"Decide that } \mathsf{H}_1 \text{ is the true hypothesis if } \mathbb{X} > 1 + \frac{\ln\left(\frac{p_0}{p_1}\right)}{\ln\left(\frac{1-p_1}{1-p_0}\right)}.\text{"}$$

Note that there is less chance of a successful transmission on Route 1 than on Route 0, and hence large values of $\mathbb{X}$ *should* lead to the decision that Route 1 was used.

(b) The MAP decision rule decides that H_1 is the true hypothesis if the likelihood ratio exceeds the threshold π_0/π_1. Now $\Lambda(1) = p_1/p_0 < 1$. Since $1-p_1 > 1-p_0$, we see that

$$\Lambda(n) = \frac{p_1(1-p_1)^{n-1}}{p_0(1-p_0)^{n-1}} = \frac{p_1(1-p_1)^{n-2}}{p_0(1-p_0)^{n-2}}\left(\frac{1-p_1}{1-p_0}\right) = \Lambda(n-1)\left(\frac{1-p_1}{1-p_0}\right) > \Lambda(n-1),$$

and so $\Lambda(1) = p_1/p_0$ is the smallest value of the likelihood ratio. It follows that if $\pi_0/\pi_1 = \pi_0/(1-\pi_0) < p_1/p_0$, that is, if $\pi_0 < p_1/(p_0+p_1)$, the MAP decision rule will always decide that H_1 is the true hypothesis regardless of the observed value of $\mathbb{X}$.

On the other hand, since $\Lambda(n)$ increases monotonically without bound as n increases, there is *no* value of $\pi_0 < 1$ for which π_0/π_1 can be guaranteed to be larger than the likelihood ratio no matter what value $\mathbb{X}$ takes on.

2. []

(a) The maximum-likelihood decision rule is indicated by the shading in the likelihood matrix shown below.

Hypothesis	$\mathbb{X}=3$	$\mathbb{X}=6$	$\mathbb{X}=9$	$\mathbb{X}=12$
H_0: excellent	0.03	0.07	0.15	0.75
H_1: good	0.10	0.15	0.60	0.15
H_2: average	0.20	0.65	0.1	0.05

(b) $P(\text{excellent student is labeled as good}) = P(\mathbb{X}=9|\mathsf{H}_0) = 0.15$.
$P(\text{excellent student is labeled as average}) = P(\{\mathbb{X}=6\}\cup\{\mathbb{X}=3\}|\mathsf{H}_0) = 0.03 + 0.07 = 0.1$.
$P(\text{average student is labeled as good or excellent }) = P(\{\mathbb{X}=9\}\cup\{\mathbb{X}=12\}|\mathsf{H}_2) = 0.15$.

(c) The conditional error probabilities of the maximum-likelihood decision rule are $P(E|\mathsf{H}_0) = 0.25$, $P(E|\mathsf{H}_1) = 0.4$, $P(E|\mathsf{H}_02) = 0.15$. Hence, the error probability is

$$P(E) = P(E|\mathsf{H}_0)P(\mathsf{H}_0) + P(E|\mathsf{H}_1)P(\mathsf{H}_1) + P(E|\mathsf{H}_2)P(\mathsf{H}_2) = 0.05 + 0.22 + .0375 = 0.3075.$$

(d) The joint probability matrix is as shown below together with the MAP decision rule.

Hypothesis	$\mathbb{X}=3$	$\mathbb{X}=6$	$\mathbb{X}=9$	$\mathbb{X}=12$
H_0: excellent	0.006	0.0140	0.030	0.1500
H_1: good	0.055	0.0825	0.330	0.0825
H_2: average	0.050	0.1625	0.025	0.0125

$P(E) = 1 - (0.15 + 0.33 + 0.1625 + 0.055) = 0.3025$ which is slightly smaller than that of the maximum-likelihood rule.

3. []

Let $C = (A \cup B)^c$ denote the event that *neither* A *nor* B occurs on a trial of the experiment, and notice that *one* of the three events A, B, and C always occurs on a trial. Then, A occurs before B does if for $n = 1, 2, 3, \ldots$, C occurs on the 1st, 2nd, $\ldots$, $(n-1)$-th trials and A occurs on the n-th trial. By independence of the trials,

$$P\{A \text{ before } B\} = P(A) + P(C)P(A) + [P(C)]^2P(A) + \cdots = P(A) \cdot \frac{1}{1 - P(C)} = \frac{P(A)}{P(A) + P(B)}.$$

Obviously, $P\{B \text{ before } A\} = 1 - P\{A \text{ before } B\} = \frac{P(B)}{P(A)+P(B)}$. The way to think about this is we can ignore all trials on which C occurs. On the very first trial on which *one* of A and B occurs, what are the chances that A occurs? Obviously $P(A \mid (A \cup B)) = \frac{P(A)}{P(A \cup B)} = \frac{P(A)}{P(A)+P(B)}$.

4. [**Network capacity and failure**]

   (a) The probability of success is the probability that one of the first two parallel links will function AND the middle link will function AND one of the last two parallel links will function: $P(\text{success}) = (p + p - p^2)(p)(p + p - p^2) = (\frac{3}{4})(\frac{1}{2})(\frac{3}{4}) = \frac{9}{32}$.

   (b) If a message can be sent, then 10 messages can be sent (any path through the network has capacity at least 10). More than 10 messages can never be sent, since the capacity of the middle link is only 10. Thus, $\mathbb{X}$ has two possible values, and using the results of part (a) we have $p_{\mathbb{X}}(10) = P(\text{success}) = \frac{9}{32}$, $p_{\mathbb{X}}(0) = P(\text{failure}) = 1 - \frac{9}{32} = \frac{23}{32}$

   (c) Simply use the definition of expectation $\mathsf{E}[\mathbb{X}] = (0)(p_{\mathbb{X}}(0)) + (10)(p_{\mathbb{X}}(10)) = \frac{90}{32} = \frac{45}{16}$.

5. [**Reliability of three $s-t$ networks**]

   (a) Network outage occurs if and only if at least one link on the upper $s-t$ path fails and at least one link on the lower $s-t$ path fails. The probability that at least one link on the upper path fails is one minus the probability that no links on the path fail, or $1-(1-p)^3 = 3p-3p^2+p^3$. The same is true for the lower path, and the states of the two paths are independent. So $P(F) = (3p - 3p^2 + p^3)^2$. For $p = 0.001$, $P(F) \approx 0.0000089820$.

   (b) $P(D_i) = \binom{4}{i}p^i(1-p)^{4-i}$ for $i = 0, 1, 3, 4$, because the number of links from among $\{1, 2, 3, 4\}$ that fail has the binomial distribution with parameters 4 and p. $P(D_{2,s}) = 2p^2(1-p)^2$ because $D_{2,s}$ is the union of two disjoint events, each having probability $p^2(1-p)^2$. The first of these events is that links 1 and 2 fail and links 3 and 4 do not fail, and the second of these events is that links 3 and 4 fail and links 1 and 2 do not fail. Similarly, $P(D_{2,d}) = 4p^2(1-p)^2$, because $D_{2,d}$ is the union of four disjoint events with probability $p^2(1-p)^2$ each. The first of these four events, for example, is that links 1 and 3 fail and links 2 and 4 do not fail.

   Next, we note that if D_0 is true, the network is equivalent to one with four parallel links directly connecting s to t, so that $P(F|D_0) = p^4$. If D_1 is true, the network is equivalent to one with two parallel links directly connecting s to t, so that $P(F|D_1) = p^2$. If $D_{2,d}$ is true, the network is equivalent to one with one link connecting s to t, so that $P(F|D_{2,d}) = p$. Finally, $P(F|D_{2,s}) = P(F|D_3) = P(F|D_4) = 1$.

   Combining the above calculations, using the law of total probability, yields

   $$P(F) = (1-p)^4p^4 + 4p(1-p)^3p^2 + 4p^2(1-p)^2p + 2p^2(1-p)^2 + 4p^3(1-p) + p^4.$$

   As a polynomial in p, the terms of $P(F)$ with the lowest powers of p are $2p^2+8p^3$. The approximate numerical value for $p = 0.01$ is $P(F) \approx 0.000208$.

   (c) Since $D_{2,s}F = D_{2,s}$ and $P(D_{2,s}) = 2p^2(1-p)^2 = 0.000001990$, the definition of conditional probabilities and the calculations above yield $P(D_{2,s}|F) = \frac{P(D_{2,s})}{P(F)} = 0.9940$. That is, given network outage occurred, the conditional probability that either links 1 and 2 both failed, or links 3 and 4 both failed, is over 99%.