ECE 313: Problem Set 3: Solutions
Conditional probabilities, independence, and the binomial distribution

1. [**Conditional probability and independence**]

   (a) All dice are equally likely to be chosen and, for a given die, all faces are equally likely. So, one can think of the event of red coming up in the first roll, denoted by R_1, as chosing one of the six red faces out of the eighteen faces in the three dice. Hence,
   $P(R_1) = \frac{6}{18} = \frac{1}{3}$.
   Another way to solve this is to use conditional probability as follows. Let D_i be the event that the die with i red faces is chosen. Recall that $\Omega = D_i \cup D_i^c$ and that D_i and D_i^c are mutually exclusive. Hence,
   $P(R_1) = P(\Omega R_1) = P((D_1 \cup D_1^c)R_1) = P(D_1R_1 \cup D_1^cR_1) = P(D_1R_1) + P(D_1^cR_1)$.
   Similarly,
   $P(D_1^cR_1) = P((D_2 \cup D_2^c)D_1^cR_1) = P(D_2D_1^cR_1) + P(D_2^cD_1^cR_1) = P(D_2R_1) + P(D_2^cD_1^cR_1)$
   because also $D_2D_1^c = D_2$. Also,
   $P(D_2^cD_1^cR_1) = P((D_3 \cup D_3^c)D_2^cD_1^cR_1) = P(D_3D_2^cD_1^cR_1) + P(D_3^cD_2^cD_1^cR_1) = P(D_3R_1)$
   because $D_3D_2^cD_1^c = D_3$ and $P(D_3^cD_2^cD_1^cR_1) = 0$. Hence,
   $P(R_1) = P(D_1R_1)+P(D_2R_1)+P(D_3R_1) = P(R_1|D_1)P(D_1)+P(R_1|D_2)P(D_2)+P(R_1|D_3)P(D_3) = \frac{1}{6}\frac{1}{3} + \frac{2}{6}\frac{1}{3} + \frac{3}{6}\frac{1}{3} = \frac{6}{18} = \frac{1}{3}$
   because $P(D_iR_1) = P(R_1|D_i)P(D_i)$.

   (b) Let R_2 be the event of red coming up in the second roll, then $P(R_2|R_1) = \frac{P(R_2R_1)}{P(R_1)}$.
   To calculate $P(R_2R_1)$, we follow a similar conditional probability approach to that of part (a) with event R_1 replace by R_2R_1:
   $P(R_2R_1) = P(D_1R_2R_1)+P(D_2R_2R_1)+P(D_3R_2R_1) = P(R_2R_1|D_1)P(D_1)+P(R_2R_1|D_2)P(D_2)+ P(R_2R_1|D_3)P(D_3) = \frac{1}{6}\frac{1}{6}\frac{1}{3} + \frac{2}{6}\frac{2}{6}\frac{1}{3} + \frac{3}{6}\frac{3}{6}\frac{1}{3} = \frac{14}{108} = \frac{7}{54}$
   because once the die is chosen, the outcomes of the rolls are independent of each other. Hence
   $P(R_2|R_1) = \frac{P(R_2R_1)}{P(R_1)} = \frac{\frac{7}{54}}{\frac{1}{3}} = \frac{7}{18}$.

2. [**Conditional probability and independence**]

   (a) There are $6^2 = 36$ possible outcomes if two dice are rolled. For event $\{R_1 = j\}$ to occur, either the first die is higher (and equal to j), in which case the second die has to be less than or equal to j (that is, it belongs to $\{1, 2, \ldots, j\}$); or the second die is higher, in which case the first die has to be less than or equal to j. Notice that the case where both dice are equal to j is double counted. Therefore,
   $p_{R_1}(j) = \frac{2j-1}{36}$, for $j \in \{1, 2, 3, 4, 5, 6\}$.
   Another way of calculating this is to use the fact that $P\{R_1 = j\} = P\{R_1 \le j\} - P\{R_1 \le j - 1\}$. For event $\{R_1 \le j\}$ to occur, both dice have to be less than or equal to j, so $P\{R_1 \le j\} = \frac{j^2}{6^2}$, and hence $P\{R_1 = j\} = P\{R_1 \le j\} - P\{R_1 \le j - 1\} = \frac{j^2}{6^2} - \frac{(j-1)^2}{6^2} = \frac{2j-1}{36}$.

   (b) Similarly, for event $\{R_2 = j\}$ to occur, either the first die is lower (and equal to j), in which case the second die has to be greater than or equal to j (that is, it belongs to $\{j, j+1, \ldots, 6\}$); or the second die is lower, in which case the first die has to be greater than or equal to j. Notice that the case where both dice are equal to j is double counted. Therefore,
   $p_{R_2}(j) = \frac{2(6-j+1)-1}{36} = \frac{13-2j}{36}$, for $j \in \{1, 2, 3, 4, 5, 6\}$.
   Another way of calculating this is to use the fact that $P\{R_2 = j\} = P\{R_2 \ge j\} - P\{R_2 \ge j + 1\}$. For event $\{R_2 \ge j\}$ to occur, both dice have to be greater than or equal to j, so $P\{R_2 \ge j\} = \frac{(6-j+1)^2}{6^2}$, and hence $P\{R_2 = j\} = P\{R_2 \ge j\} - P\{R_2 \ge j+1\} = \frac{(6-j+1)^2}{6^2} - \frac{(6-(j+1)+1)^2}{6^2} = \frac{13-2j}{36}$.

(c) One can use the fact that $P\{W_1 = j\} = P\{W_1 \le j\} - P\{W_1 \le j-1\}$ for $j \in \{1, \ldots, 6\}$. For event $\{W_1 \le j\}$ to occur, all three dice have to be less than or equal to j, so $P\{W_1 \le j\} = \frac{j^3}{6^3}$, and hence $P\{W_1 = j\} = P\{W_1 \le j\} - P\{W_1 \le j-1\} = \frac{j^3}{6^3} - \frac{(j-1)^3}{6^3} = \frac{3j^2-3j-1}{216}$.

(d) There are two outcomes such that event $\{R_1 = 5, R_2 = 4\}$ occurs: either die one shows 5 and die two shows 4 or viceversa. Therefore, $P\{R_1 = 5, R_2 = 4\} = \frac{2}{36} = \frac{1}{18}$. However, $P\{R_1 = 5\}P\{R_2 = 4\} = \frac{9}{36}\frac{5}{36} \ne P\{R_1 = 5, R_2 = 4\}$. Therefore, the events $\{R_1 = 5\}$ and $\{R_2 = 4\}$ are not independent.

(e) The outcomes of rolling the red dice do not affect the outcomes of rolling the white dice so one can use the principle of counting to obtain that $P\{R_1 = 5, W_2 = 2\} = \frac{(|\{R_1=5\}|)(|\{W_2=2\}|)}{|\Omega|} = \frac{(|\{R_1=5\}|)(|\{W_2=2\}|)}{6^5} = \frac{|\{R_1=5\}|}{6^2}\frac{|\{W_2=5\}|}{6^3} = P\{R_1 = 5\}P\{W_2 = 2\}$, and hence the events $\{R_1 = 5\}$ and $\{W_2 = 2\}$ are independent.

(f) $P\{R_2 = 2|R_1 = 3\} = \frac{P\{R_2=2,R_1=3\}}{P\{R_1=3\}} = \frac{2/36}{5/36} = \frac{2}{5}$, because there are two outcomes such that event $\{R_2 = 2, R_1 = 3\}$ occurs: either die one shows 2 and die two shows 3 or viceversa; and also $P\{R_1 = j\} = \frac{2j-1}{36}$.

3. **[Independence]**

(a) Let R_i be the event that the i-th spin lands on red. Then $P\{X > 3\} = P(R_1 \cup R_1^c R_2 R_3) = P(R_1) + P(R_1^c)P(R_2)P(R_3) = \frac{18}{38} + \left(1 - \frac{18}{38}\right)\frac{18}{38}\frac{18}{38} \approx 0.5918$ because events R_1 and $R_1^c R_2 R_3$ are mutually exclusive, and also events R_1^c, R_2, R_3 are independent.

(b) First, $X \in \{4, 2, 0\}$. Then,

$$
\begin{aligned}
p_X(4) &= P(R_1 \cup R_1^c R_2 R_3) = \frac{18}{38} + \left(1 - \frac{18}{38}\right)\frac{18}{38}\frac{18}{38} \\
p_X(2) &= P(R_1^c R_2 R_3^c \cup R_1^c R_2^c R_3) = \left(1 - \frac{18}{38}\right)\frac{18}{38}\left(1 - \frac{18}{38}\right) + \left(1 - \frac{18}{38}\right)\left(1 - \frac{18}{38}\right)\frac{18}{38} \\
p_X(0) &= P(R_1^c R_2^c R_3^c) = \left(1 - \frac{18}{38}\right)\left(1 - \frac{18}{38}\right)\left(1 - \frac{18}{38}\right)
\end{aligned}
$$

Hence $E[X] = \sum_i u_i p_X(u_i) = (4)p_X(4) + (2)p_X(2) + (0)p_X(0) \approx 2.892$.

(c) Even though the probability of winning ($X > 3$) is almost 0.6, on average you loose almost 11 cents.

4. **[Conditional probability and the binomial distribution]**

(a)

$$
\begin{aligned}
P\{X \le 5|X \ge 4\} &= \frac{P\{X \le 5, X \ge 4\}}{P\{X \ge 4\}} = \frac{P\{X \in \{4,5\}\}}{1 - P\{X < 4\}} \\
&= \frac{\binom{10}{4}\left(\frac{1}{2}\right)^{10} + \binom{10}{5}\left(\frac{1}{2}\right)^{10}}{1 - \left[\binom{10}{0}\left(\frac{1}{2}\right)^{10} + \binom{10}{1}\left(\frac{1}{2}\right)^{10} + \binom{10}{2}\left(\frac{1}{2}\right)^{10} + \binom{10}{3}\left(\frac{1}{2}\right)^{10}\right]} \\
&= \frac{231}{424} \approx 0.5448
\end{aligned}
$$

because X is a binomial random variable with parameters $n = 10$ and $p = 1/2$.

(b) Let H_4 denote the event that the fourth toss shows heads, and let $\hat{X}$ be the number of heads observed in the other nine tosses. Then, the events H_4 and $\{\hat{X} = 3\}$ are independent (because the tosses are independent), and $\hat{X}$ is a binomial random variable with parameters $n = 9$ and

$p = 1/2$. Therefore,

$$P\{H_4|X=4\} = \frac{P\{H_4, X=4\}}{P\{X=4\}} = \frac{P\{H_4, \hat{X}=3\}}{P\{X=4\}} = \frac{P(H_4)P\{\hat{X}=3\}}{P\{X=4\}} = \frac{\frac{1}{2}\left(\binom{9}{3}\left(\frac{1}{2}\right)^9\right)}{\binom{10}{4}\left(\frac{1}{2}\right)^{10}} = \frac{2}{5}$$

(c) Similarly to part (b), but with the probability of heads being equal to p instead of $1/2$,

$$P\{H_4|X=4\} = \frac{P\{H_4, X=4\}}{P\{X=4\}} = \frac{P\{H_4, \hat{X}=3\}}{P\{X=4\}} = \frac{P(H_4)P\{\hat{X}=3\}}{P\{X=4\}} = \frac{p\left(\binom{9}{3}p^3(1-p)^{9-3}\right)}{\binom{10}{4}p^4(1-p)^{10-4}} = \frac{2}{5}.$$

Thus, not knowing p does not disadvantage you; the probability is $2/5$ regardless of the value of p.

5. **[Binomial distribution]**

(a) Notice that doubling and halving are inverse functions of each other, so halving can be considered as doubling -1 times. If you let n be the number of weeks in which your money halved, then you'll be left with $\$32\left(2^{5-2n}\right)$ at the end of the five weeks, where $n \in \{0, 1, 2, 3, 4, 5\}$. Therefore, $X \in \{1024, 256, 64, 16, 4, 1\}$.

(b) If you let Y be a random variable denoting the number of weeks in which your money halved, then $X = \$32\left(2^{5-2Y}\right) = 4^{5-Y}$. Hence,

$p_X\left(4^{5-n}\right) = \binom{5}{n}\left(\frac{1}{2}\right)^5$, for $n = 0, 1, 2, 3, 4, 5$.

(c) $E[X] = \sum_i u_i p_X(u_i) = \left(4^{5-n}\right)\binom{5}{n}\left(\frac{1}{2}\right)^5 \approx 97.66$.

The expected amount of money you'll have at the end of the five weeks is just over three times as much as what you started with, so the TV add is accurate.

(d) Let L be the event of loosing money after five weeks, then

$P(L) = P\{X \in \{16, 4, 1\}\} = p_X(16) + p_X(4) + p_X(1) = 1/2$.