ECE 313: Final Exam

1. (a) One way to solve this is using conditional probability using the fact that coin C_3 is grabbed among three coins, while coin C_1 is then only grabbed among two coins. Therefore,
$P(C_3C_1) = P(C_1|C_3)P(C_3) = \frac{1}{2}\frac{1}{3}\ \boxed{= \frac{1}{6}.}$

(b) One way to solve this is using conditional probability using two facts: tails showing on the first flip only depends on which coin was grabbed first so that $P(T_1|C_3C_1) = P(T|C_3)$; and heads showing on the second flip only depends on which coin was grabbed second (because of the conditioning on coin C_1 being grabbed second) so that $P(H_2|C_3C_1T_1) = P(H|C_1)$. Therefore,
$P(T_1H_2|C_3C_1) = P(H_2|C_3C_1T_1)P(T_1|C_3C_1) = P(H|C_1)P(T|C_3) = \frac{1}{4}\frac{2}{3}\ \boxed{= \frac{1}{6}.}$

(c) X can take values in $\{0, 1, 2\}$. One way to solve this is using total probability by partitioning the sample space Ω into the six events representing the sequence of coins chosen, using the same notation as in parts (a) and (b) and also using the results therein.

$$\begin{aligned} P\{X=0\} &= P(T_1T_2|C_1C_2)P(C_1C_2) + P(T_1T_2|C_1C_3)P(C_1C_3) \\ &\quad + P(T_1T_2|C_2C_1)P(C_2C_1) + P(T_1T_2|C_2C_3)P(C_2C_3) \\ &\quad + P(T_1T_2|C_3C_1)P(C_3C_1) + P(T_1T_2|C_3C_2)P(C_3C_2) \\ &= \frac{3}{4}\frac{3}{4}\frac{1}{6} + \frac{3}{4}\frac{2}{3}\frac{1}{6} + \frac{3}{4}\frac{3}{4}\frac{1}{6} + \frac{3}{4}\frac{2}{3}\frac{1}{6} + \frac{2}{3}\frac{3}{4}\frac{1}{6} + \frac{2}{3}\frac{3}{4}\frac{1}{6} = \frac{25}{48}. \\ P\{X=2\} &= P(H_1H_2|C_1C_2)P(C_1C_2) + P(H_1H_2|C_1C_3)P(C_1C_3) \\ &\quad + P(H_1H_2|C_2C_1)P(C_2C_1) + P(H_1H_2|C_2C_3)P(C_2C_3) \\ &\quad + P(H_1H_2|C_3C_1)P(C_3C_1) + P(H_1H_2|C_3C_2)P(C_3C_2) \\ &= \frac{1}{4}\frac{1}{4}\frac{1}{6} + \frac{1}{4}\frac{1}{3}\frac{1}{6} + \frac{1}{4}\frac{1}{4}\frac{1}{6} + \frac{1}{4}\frac{1}{3}\frac{1}{6} + \frac{1}{3}\frac{1}{4}\frac{1}{6} + \frac{1}{3}\frac{1}{4}\frac{1}{6} = \frac{11}{144}. \\ P\{X=1\} &= 1 - P\{X=0\} - P\{X=2\} = \frac{29}{72}. \end{aligned}$$

Therefore, $\boxed{p_X(i) = \begin{cases} \frac{25}{48} & i = 0, \\ \frac{29}{72} & i = 1, \\ \frac{11}{144} & i = 2, \\ 0 & else. \end{cases}}$

2. (a) f is negative in that interval, g is the pdf of a uniform in that interval, and h is the pdf of an exponential with parameter $\lambda = 2$. Therefore, $\boxed{g \text{ and } h \text{ are valid pdfs.}}$

(b) f is the product of the pdfs of an exponential with parameter $\lambda = 3$ and a standard normal. g is negative on that interval for x. h is the product of the pdfs of a uniform on $(1/4, 3/4)$ and another uniform on $(0, 1)$. Therefore, $\boxed{f \text{ and } h \text{ are valid joint pdfs.}}$

3. (a) The joinf pdf is the product of the pdfs of an exponential with parameter $\lambda = 1$ and a uniform on $(0, 2)$. Therefore, $\boxed{f_X(u) = e^{-u} \text{ for } u \geq 0 \text{ and zero else.}}$

(b) X and Y are independent. Therefore, $\boxed{f_{X|Y}(u|v) = e^{-u} \text{ for } u \geq 0 \text{ and zero else.}}$

(c) X and Y are independent so the pdf of Z can be obtained through convolution of the pdfs of X and Y.
For $\alpha \in (0, 2)$, $f_Z(\alpha) = \int_0^\alpha \frac{1}{2}e^{u-\alpha}du = \frac{1}{2}(1 - e^{-\alpha})$.
For $\alpha \geq 2$, $f_Z(\alpha) = \int_0^2 \frac{1}{2}e^{u-\alpha}du = \frac{e^{-\alpha}}{2}(e^2 - 1)$.

Therefore, $\boxed{f_Z(\alpha) = \begin{cases} 0 & \alpha \leq 0, \\ \frac{1}{2}(1 - e^{-\alpha}) & \alpha \in (0, 2), \\ \frac{e^{-\alpha}}{2}(e^2 - 1) & \alpha \geq 2. \end{cases}}$

4. (a) Y is the sum of two independent binomial random variables, one being the number of heads that show from flipping C_2 4 times, and the other being the number of heads that show from flipping C_3 3 times. Both coins have the same probability of heads, therefore Y is a binomial random variable with parameters $n = 4 + 3 = 7$ and $p = 1/4$. That is,

$$\boxed{p_Y(i) = \binom{7}{i} \left(\tfrac{1}{4}\right)^i \left(\tfrac{3}{4}\right)^{7-i} \text{ for } i = 0, 1, \ldots, 7 \text{ and zero else.}}$$

(b) Z is simply the sum of N Bernoulli random variables X_i with parameter $p = 1/3$, each having mean $1/3$. One can obtain the result by conditioning on the value of N, which has mean $7/2$. That is,

$$\begin{aligned}
\mathsf{E}[Z] &= E\left[\sum_{i=1}^{N} X_i\right] = \sum_{n=1}^{6} E\left[\sum_{i=1}^{N} X_i | N = n\right] P\{N = n\} \\
&= \sum_{n=1}^{6} E\left[\sum_{i=1}^{n} X_i\right] P\{N = n\} = \sum_{n=1}^{6} \sum_{i=1}^{n} E\left[X_i\right] P\{N = n\} \\
&= \sum_{n=1}^{6} n E\left[X_1\right] P\{N = n\} = E\left[X_1\right] \sum_{n=1}^{6} n P\{N = n\} = E\left[X_1\right] E\left[N\right] = \tfrac{1}{3}\tfrac{7}{2} \\
&= \tfrac{7}{6}
\end{aligned}$$

Therefore, $\boxed{\mathsf{E}[Z] = \tfrac{7}{6}.}$

5. (a) Let S be the event a signal is transmitted. By the law of total probability,

$$\begin{aligned}
f_{\mathbb{X}}(u) &= f_{\mathbb{X}|S}(u|S)P[S] + f_{\mathbb{X}|S^c}(u|S^c)P[S^c] \\
&= \begin{cases} \frac{1}{16} & -2 \le u \le 0 \\ \frac{5}{16} & 0 < u \le 2 \\ \frac{1}{4} & 2 < u \le 3 \\ 0 & \text{else.} \end{cases}
\end{aligned}$$

(b) By Bayes rule,

$$\begin{aligned}
P[S|X = u] &= \frac{f_{\mathbb{X}|S}(u|S)P[S]}{f_{\mathbb{X}}(u)} \\
&= \frac{\frac{1}{3} \cdot \frac{3}{4}}{\frac{5}{16}} \\
&= \frac{4}{5}.
\end{aligned}$$

6. (a) Since the experiment is under a uniform probability distribution, we can solve this problem by counting. We want to count the number of ways to catch x goldfish in a total of k fish caught. We first select x goldfish and then select $k - x$ catfish, completely counting the sets of size k containing exactly x goldfish. The number of ways to select x goldfish in the first step is given by $\binom{g}{x}$, and the number of ways to select $k - x$ catfish is given by $\binom{c}{k-x}$ and so there are $\binom{g}{x} \cdot \binom{c}{k-x}$ ways of selecting a catch k with x goldfish. We normalize by the total number of ways to catch k fish, to get the solution:

$$\frac{\binom{g}{x} \cdot \binom{c}{k-x}}{\binom{g+c}{k}}.$$

(b) Again, we use counting to solve for the probability. We decompose the counting problem into two steps: we first select the two goldfish to be caught again from our original catch, and then we select the $m-2$ other fish from the $g+c-x$ fish that were not goldfish caught the first time. Therefore we have $\binom{x}{2}\binom{g+c-x}{m-2}$, giving us the probability of

$$\frac{\binom{x}{2}\binom{g+c-x}{m-2}}{\binom{g+c}{m}}.$$

7. (a) Using the law of total probability, the solution is $\frac{1}{4}\frac{1}{2}+\frac{1}{16}\frac{1}{2}=\frac{5}{32}$.

(b) Using Bayes rule, the solution is

$$\frac{\frac{1}{16}\frac{1}{2}}{\frac{1}{4}\frac{1}{2}+\frac{1}{16}\frac{1}{2}}=\frac{1}{5}.$$

(c) First we see that $\mathbb{X}$ is equally likely to take on the values 1 or 2. The range of $\mathbb{Y}$ is $\{0,1,2\}$ and the pmf of $\mathbb{Y}$ can be found by conditioning on the two values of $\mathbb{X}$. Using the law of total probability, we get

$$\begin{aligned} p_{\mathbb{Y}}(0) &= \frac{1}{4}\frac{1}{2}+\frac{1}{16}\frac{1}{2}=\frac{5}{32} \\ p_{\mathbb{Y}}(1) &= \frac{3}{4}\frac{1}{2}+2\frac{3}{4}\frac{1}{4}\frac{1}{2}=\frac{9}{16} \\ p_{\mathbb{Y}}(2) &= 0\frac{1}{2}+\left(\frac{3}{4}\right)^2\frac{1}{2}=\frac{9}{32}. \end{aligned}$$

8. (a) The expected time before the Ray breaks is

$$\begin{aligned} \mathsf{E}[\mathbb{X}|\mathbb{X}>u] &= u+\int_0^\infty P\{\mathbb{X}>u+v|\mathbb{X}>u\}\,dv \\ &= u+\int_0^\infty P\{\mathbb{X}>v\}\,dv \\ &= u+\frac{1}{\lambda_x}. \end{aligned}$$

where we used the memoryless property of the exponential r.v. in the second step.

(b) The probability that a Ray breaks before a Spring is given by

$$\begin{aligned} P\{\mathbb{X}<\mathbb{Y}\} &= \int_0^\infty\int_0^v \lambda_x e^{-\lambda_x u}\lambda_y e^{-\lambda_y v}\,du\,dv \\ &= \int_0^\infty (1-e^{-\lambda_x v})\lambda_y e^{-\lambda_y v}\,dv \\ &= \frac{\lambda_x}{\lambda_x+\lambda_y}. \end{aligned}$$

9. (a) This event occurs if and only if the $(N+1)$th choice of the left-hand matchbox is made at the $(N+1+N-k)$th trial. Recall the negative binomial distribution, we see that

$$P_k=\binom{2N-k}{N}p^{N+1}(1-p)^{N-k}.$$

Hence, $\boxed{M=2N-k.}$.

(b) The maximum likelihood estimate, $\hat{p}_{ML}$, is the value of p that maximizes $\binom{2N-j}{N}p^{N+1}(1-p)^{N-j}$. Equivalently, $\hat{p}_{ML}$ is the value that maximizes $p^{N+1}(1-p)^{N-j}$. First, assume that $j \le N-1$.

$$\begin{aligned}\frac{d(p^{N+1}(1-p)^{N-j})}{dp} &= [(N+1)(1-p)-p(N-j)]p^N(1-p)^{N-j-1} \\ &= [N+1-(2N+1-j)p]p^N(1-p)^{N-j-1}.\end{aligned}$$

The derivative is positive if $p < \frac{N+1}{2N+1-j}$ and negative if $p > \frac{N+1}{2N+1-j}$. Hence the likelihood is maximized at $\boxed{p = \frac{N+1}{2N+1-j}}$.

If $j = N$, then the likelihood is p^{N+1}, which is maximized at $p = 1$, so the formula $\hat{p}_{ML}(j) = \frac{N+1}{2N+1-j}$ is true for $0 \le j \le N$.

10. (a)

$$P(\text{ no arrival in the interval } [0,2]) = e^{-2\lambda}.$$

Hence, the answer is $\boxed{e^{-2\lambda}}$.

(b) The number of arrivals in the interval $[1,3]$ is a Poisson random variable with mean 2λ, hence the probability is $\boxed{e^{-2\lambda}\left(1+2\lambda+\frac{(2\lambda)^2}{2}\right).}$

(c) Let N be the number of arrivals during $[0,3]$, N_1 be the number of arrivals during $[0,1]$ and N_2 be the number of arrivals during $[1,3]$.

$$\begin{aligned}P(N_1 = 2|N = 3) &= \frac{P(N_1 = 2, N_2 = 1)}{P(N = 3)} = \frac{P(N_1 = 2)P(N_2 = 1)}{P(N = 3)} \\ &= \frac{e^{-\lambda}\lambda^2}{2!}e^{-2\lambda}(2\lambda)\frac{3!}{e^{-3\lambda}(3\lambda)^3} \\ &= \frac{3!\cdot 2}{2!\cdot 27} = \frac{2}{9}.\end{aligned}$$

Hence, the answer is $\boxed{\frac{2}{9}}$.

11. (a) Observe that Z is a Gaussian random variable with mean 2 and variance 9.

$$\begin{aligned}P(Z < 0.5) &= P\left(\frac{Z-2}{3} < \frac{0.5-2}{3}\right) \\ &= \Phi(-0.5) = 1-\Phi(0.5) = 1-0.6915 = 0.3085\end{aligned}$$

from the normal tables supplied.
Hence $\boxed{P(Z < 0.5) = 0.3085.}$

(b) $Cov(X,Z) = Cov(X, X+Y) = Var(X) = 4$.
$Var(Z) = Var(X+Y) = Var(X)+Var(Y) = 4+5 = 9$.
$E(Z) = E(X+Y) = 2+0 = 2$.

$$\begin{aligned}L^*(Z) &= \mu_X + \frac{Cov(X,Z)}{Var(Z)}(Z-E(Z)) \\ &= 2+\frac{4}{9}(Z-2) = 2+\frac{4(Z-2)}{9},\end{aligned}$$

Hence, $\boxed{L^*(Z) = 2 + \frac{4(Z-2)}{9}.}$

The best linear estimator happens to coincides with the best unconstrained estimator in this case (see 4.11 in the notes).

12. (a) TRUE.

$P(A|B)P(B) + P(A^c|B)P(B) = P(A)$ is a FALSE statement.

FALSE. It could be that $P(A \cap B) = 0$.

(b) FALSE. These are conditional probabilities and could both be zero.

FALSE. The prior probability π_0 could be very small leading to this scenario.

TRUE. An important property of the MAP detection rule.

TRUE. When the prior probabilities are equal, the MAP and ML rule decision rules coincide.

6.4 Normal tables

Tables 6.1 and 6.2 below were computed using Abramowitz and Stegun, *Handbook of Mathematical Functions*, Formula 7.1.26, which has maximum error at most 1.5×10^{-7}.

Table 6.1: Φ function, the area under the standard normal pdf to the left of x.

x	**0.00**	**0.01**	**0.02**	**0.03**	**0.04**	**0.05**	**0.06**	**0.07**	**0.08**	**0.09**
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

Table 6.2: Q function, the area under the standard normal pdf to the right of x.

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641
0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010

x	0.0	0.2	0.4	0.6	0.8
0.0	0.5000000	0.4207403	0.3445783	0.2742531	0.2118553
1.0	0.1586553	0.1150697	0.0807567	0.0547993	0.0359303
2.0	0.0227501	0.0139034	0.0081975	0.0046612	0.0025552
3.0	0.0013500	0.0006872	0.0003370	0.0001591	0.0000724
4.0	0.0000317	0.0000134	0.0000054	0.0000021	0.0000008