ECE 313: Problem Set 12
Joint pdfs, Covariance, LLN and CLT

Due:	Wednesday, December 1, at 4 p.m.
Reading:	*ECE 313 Notes* Sections 4.7-4.9.

1. **[Joint pdfs of functions of random variables]**
Digital integrated circuits designed in nanoscale process technologies exhibit a dynamic power dissipation P_D and delay T_d given by:

$$P_D = k_1 V_{dd}(V_{dd} - V_t)$$
$$T_d = \frac{k_2 V_{dd}}{(V_{dd} - V_t)}$$

where V_t is the device threshold voltage, V_{dd} is the supply voltage, and k_1 and k_2 are constants. The supply voltage V_{dd} and the threshold voltage V_t are independent random variables. Thus, the power dissipation and the delay of nanoscale integrated circuits are random variables as well. In this problem, you are asked to determine the joint pdf of P_D and T. Let X, Y, W and Z, denote the random variables V_{dd}, V_t, P_D and T_d, respectively. Now answer the following:

   (a) The supply voltage is uniformly distributed in the range $[0.8V, 1.2V]$. The threshold voltage V_t is known to be nonnegative and has a symmetric Gaussian like pdf with a mean of $0.3V$, standard deviation of $0.1V$, and support $[0, 0.6V]$. Specifically, the pdf of V_t is a constant times the Gaussian pdf with mean $0.3V$ and standard deviation of $0.1V$, over the interval $[0, 0.6V]$, and is zero outside the interval. The constant used in the interval is chosen so that the pdf integrates to one (as in Example 3.6.6 of the notes). Determine the joint pdf, $f_{X,Y}(u, v)$.

   (b) The given relationship determines (W, Z) as a function, $g(X, Y)$, of (X, Y). View g as a function from the support of $f_{X,Y}$ in the $u-v$ plane, to the $\alpha-\beta$ plane. Is g one-to-one? If it is, determine the inverse transformation $(u, v) = g^{-1}(\alpha, \beta)$ (so that $(X, Y) = g^{-1}(W, Z)$).

   (c) Determine the pdf $f_{W,Z}(\alpha, \beta)$.
   The expression for the support of $f_{W,Z}$ (i.e. the set on which it is not zero) is complicated and omitted.

2. **[Covariance I]**
Consider random variables X and Y on the same probability space.

   (a) If $\text{Var}(X + 2Y) = 40$ and $\text{Var}(X - 2Y) = 20$, what is $\text{Cov}(X, Y)$?

   (b) In part (a), determine $\rho_{X,Y}$ if $\text{Var}(X) = 2 \cdot \text{Var}(Y)$.

   The next two parts are independent of parts (a) and (b), and of each other. In particular, the numbers from part (a) are not to be assumed.

   (c) If $\text{Var}(X + 2Y) = \text{Var}(X - 2Y)$, are X and Y uncorrelated ?

   (d) If $\text{Var}(X) = \text{Var}(Y)$, are X and Y uncorrelated?

3. **[Covariance II]**
Rewrite the expressions below in terms of $\text{Var}(X)$, $\text{Var}(Y)$, $\text{Var}(Z)$, and $\text{Cov}(X, Y)$.

   (a) $\text{Cov}(3X + 2, 5Y - 1)$

   (b) $\text{Cov}(2X + 1, X + 5Y - 1)$.

   (c) $\text{Cov}(2X + 3Z, Y + 2Z)$ where Z is uncorrelated to both X and Y.

4. **[Covariance III]**
Random variables X_1 and X_2 represent two observations of a signal corrupted by noise. They have the same mean μ and variance σ^2. The *signal-to-noise-ratio* (SNR) of the observation X_1 or X_2 is defined as the ratio $SNR_X = \frac{\mu^2}{\sigma^2}$. A system designer chooses the averaging strategy, whereby she constructs a new random variable $S = \frac{X_1+X_2}{2}$.

   (a) Show that the SNR of S is twice that of the individual observations, if X_1 and X_2 are uncorrelated.

   (b) The system designer notices that the averaging strategy is giving $SNR_S = (1.5)SNR_X$. She correctly assumes that the observations X_1 and X_2 are correlated. Determine the value of the correlation coefficient $\rho_{X_1X_2}$.

   (c) Under what condition on $\rho_{X,Y}$ can the averaging strategy result in an SNR_S that is arbitrarily high?

5. **[Law of Large Numbers and Central Limit Theorem]**
A fair die is rolled n times. Let $S_n = X_1 + X_2 + \ldots + X_n$, where X_i is the number showing on the i^{th} roll. Determine a condition on n so the probability the sample average $\frac{S_n}{n}$ is within 1% of the mean μ_X, is greater than 0.95. (Note: This problem is related to Example 4.9.2 in the notes, but the variance used in the notes is incorrect. See corrections to notes on the course website if interested.)

   (a) Solve the problem using the form of the law of large numbers based on the Chebychev inequality (i.e. Proposition 4.9.1 in the notes).

   (b) Solve the problem using the Gaussian approximation for S_n, which is suggested by the CLT. (Do not use the continuity correction, because, unless $3.5n \pm (0.01)n\mu_X$ are integers, inserting the term 0.5 is not applicable).