University of Illinois **Fall 2010**

ECE 313: Problem Set 9
Functions of a random variable, failure rate functions

Due:	Wednesday October 27 at 4 p.m.
Reading:	*ECE 313 Notes* Sections 3.8-3.10.

1. **[Log random variables]**

   (a) Let $Z = e^U$, where U is uniformly distributed over an interval $[a, b]$. Find the pdf, f_Z. Be sure to specify it over the entire real line. (The random variable Z is said to have a log uniform distribution because $\ln(Z)$ has a uniform distribution.)

   (b) Find $E[Z]$. (Hint: Use LOTUS.)

   (c) Let $Y = e^X$, where X has a normal distribution. For simplicity, suppose X has mean zero and variance one. Find the pdf, f_Y. (The random variable Y is said to have a log normal distribution because $\ln(Y)$ has a normal distribution. This distribution arises as the amplitude of a signal after propagation through a heterogeneous media such as in tomography or atmospheric propagation, where attenuation factors in different parts of the media are mutually independent.)

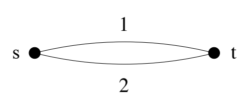
   (d) Find $E[Y]$. (Hint: Use LOTUS. Rewrite the integrand as a constant times a Gaussian pdf by completing the square in the exponent, and integrate out the pdf to get a simple answer.)

2. **[Generation of random variables with specified probability density function]**
   Find a function g so that, if U is uniformly distributed over the interval $[0, 1]$, and $X = g(U)$, then X has the pdf:

   $$f_X(v) = \begin{cases} 2v & \text{if } 0 \leq u \leq 1 \\ 0 & \text{else.} \end{cases}$$

   (Hint: Begin by finding the cumulative distribution function F_X.)

3. **[Failure rate of a network with two parallel links]**
   Consider the $s - t$ network with two parallel links, as shown:



   Suppose that for each i, link i fails at time T_i, where T_1 and T_2 are independent, exponentially distributed with some parameter $\lambda > 0$. The network fails at time T, where $T = \max\{T_1, T_2\}$.

   (a) Express $F_T^c(t) = P\{T > t\}$ for $t \geq 0$ in terms of t and λ.

   (b) Find the pdf of T.

   (c) Find the failure rate function, $h(t)$, for the network. Simplify your answer as much as possible. (Hint: Check that your expression for h satisfies $h(0) = 0$ and $\lim_{t\to\infty} h(t) = \lambda$.)

   (d) Find $P(\min\{T_1, T_2\} < t | T > t)$ and verify that $h(t) = \lambda P\{\min\{T_1, T_2\} < t | T > t)$. That is, the network failure rate at time t is λ times the conditional probability that at least one of the links has already failed by time t, given the network has not failed by time t.

4. **[Cauchy vs. Gaussian detection problem]**
On the basis of a sensor output X, it is to be decided which hypothesis is true: H_0 or H_1.
Under H_1, X is a Gaussian random variable with mean zero and variance σ^2=0.5: $f_1(u) = \frac{1}{\sqrt{\pi}}e^{-u^2}$.
Under H_0, X has the Cauchy density, $f_0(u) = \frac{1}{\pi(1+u^2)}$.

(a) Find the ML decision rule. Express it as simply as possible.

(b) Find $p_{\text{false alarm}}$ and p_{miss}. Hint: the CDFs of X under the hypotheses are $F_0(c) = 0.5 + \frac{\arctan(c)}{\pi}$ and $F_1(c) = \Phi(c\sqrt{2})$, respectively.

(c) Consider the decision rule that decides H_1 is true if $|X| \leq 1.4$ and decides H_0 otherwise. This rule is the MAP rule for some prior distribution (π_1, π_0). Find the ratio $\tau = \pi_0/\pi_1$ for that distribution.