University of Illinois **Fall 2010**

ECE 313: Problem Set 1-11: Problems and Solutions
Randomly selected problems for ECE 313 review session Friday, 11/12/10

Note:	This is a compilation of randomly selected problems. There is one problem each from problem sets 7-11, and four problems selected at random from among those on Exam II for the last six semesters of 313. These problems were selected independently of Exam II and may or may not be similar to problems on the exam. **They definitely do not cover all the material that could be covered on the exam. The purpose of the document is solely to aid in the review session led by the instructors on Friday, November 12, during class. The complete list of topics the exam is selected from is given by the table on contents of the notes, from Sections 3.1 to 4.6.**
Reminder:	Hour Exam II will be held Monday November 15, 7:00 p.m. – 8:00 p.m. Section C (10 am section) Room 124 Burrill Hall, all other sections: Room 100 Noyes Laboratory One two-sided 8.5”×11” sheet of notes allowed, with font size no smaller than 10 pt or equivalent handwriting. Bring a picture ID. The exam will cover the reading assignments, lectures, and problems associated with problem sets 1-11, with an emphasis on problem sets 7-11.

1. **[Uniform and exponential distribution, II-PS7, number 4]**
A factory produced two equal size batches of radios. All the radios look alike, but the lifetime of a radio in the first batch is uniformly distributed from zero to two years, while the lifetime of a radio in the second batch is exponentially distributed with parameter $\lambda = 0.1(\text{years})^{-1}$.

   (a) Suppose Alicia bought a radio and after five years it is still working. What is the conditional probability it will still work for at least three more years?
   Solution: Let B=radio comes from the first batch, L=lifetime of radio

$$P\{L \geq c\} = P(L \geq c|B)P(B) + P(L \geq c|B^c)P(B^c)$$

$$P(L \geq c) = \begin{cases} 1 & c \leq 0 \\ (1 - (c/2) + e^{-c/10})(1/2) & 0 \leq c \leq 2 \\ e^{-c/10}\frac{1}{2} & c \geq 2 \end{cases}$$

$$P(L \geq 5 + 3|L \geq 5) = \frac{(1/2)e^{-0.8}}{(1/2)e^{-0.5}} = e^{-0.3} = 0.7408$$

   A shorter answer is the following. Given $L \geq 5$, the battery must be from the second batch, and thus have an expoentially distributed lifetime with parameter λ. By the memoryless property of the exponential distribution, the battery after five years is as good a new. So the probability it lasts at least three more years is $e^{-3\lambda} = e^{-0.3}$.

   (b) Suppose Venkatesh bought a radio and after one year it is still working. What is the conditional probability it will work for at least three more years?
   Solution:

$$P[L \geq 1 + 3|L \geq 1] = \frac{1/2e^{-0.4}}{1/2(1/2 + e^{-0.1})} = 0.4771$$

2. **[Gaussian Distribution-ps8, number 1]**
The random variable X has a $\mathcal{N}(-4, 9)$ distribution. Determine the following probabilities:

   (a) $P\{X = 0\}$
   Solution: As X is a continuous-type random variable, $P\{X = c\} = 0$ for any value of c including $c = 0$.

(b) $P\{|X+4| \geq 2\}$
Solution:

$$\begin{aligned} P\{|X+4| \geq 2\} &= P\{X \leq -6 \cup X \geq -2\} \\ &= P\{X \leq -6\} + P\{X \geq -2\} = 2P\{X \geq -2\} \\ &= 2P\{\frac{X+4}{3} \geq 2/3\} = 2Q(2/3) = 2 \times 0.2514 = 0.5028 \end{aligned}$$

(c) $P\{0 < X < 2\}$
Solution:

$$\begin{aligned} P\{0 < X < 2\} &= P\{X > 0\} - P\{X \geq 2\} \\ &= P\{\frac{X+4}{3} > 4/3\} - P\{\frac{X+4}{3} \geq 2\} \\ &= Q(4/3) - Q(2) = 0.0.0918 - 0.0228 = 0.069 \end{aligned}$$

(d) $P\{X^2 < 9\}$
Solution:

$$\begin{aligned} P\{X^2 < 9\} &= P\{-3 < X < 3\} \\ &= P\{\frac{1}{3} < \frac{X+4}{3} < 7/3\} \\ &= \Phi(7/3) - \Phi(1/3) \approx \Phi(2.33) - \Phi(0.33) = 0.9901 - 0.6293 = 0.3608 \end{aligned}$$

3. **[Generation of random variables with specified probability density function PS 9, number 3]**
Find a function g so that, if U is uniformly distributed over the interval $[0,1]$, and $X = g(U)$, then X has the pdf:

$$f_X(v) = \begin{cases} 2v & \text{if } 0 \leq u \leq 1 \\ 0 & \text{else.} \end{cases}$$

(Hint: Begin by finding the cumulative distribution function F_X.)

Solution: Since U and X are both distributed over the interval $[0,1]$, the function g should map the interval $[0,1]$ onto interval $[0,1]$. The desired CDF of $g(U)$ is given by $F_X(c) = \int_0^c 2v dv = c^2$ for $0 \leq c \leq 1$, and we let $g(u) = F^{-1}(u)$. If $g(u) = c$ then $F_X(c) = u$ or $c^2 = u$ or $c = \sqrt{u}$. That is, $g(u) = \sqrt{u}$ for $0 \leq u \leq 1$ works.

4. **[Recognizing independence PS10-number 4]**
Decide whether X and Y are independent for each of the following three joint pdfs. If they are independent, identify the marginal pdfs f_X and f_Y. If they are not, give a reason why.

(a) $f_{X,Y}(u,v) = \begin{cases} \frac{4}{\pi} e^{-(u^2+v^2)} & u, v \geq 0 \\ 0 & \text{else.} \end{cases}$

Solution: Yes. $f_X(u) = f_Y(u) = \begin{cases} \frac{2}{\sqrt{\pi}} e^{-u^2} & u \geq 0 \\ 0 & \text{else.} \end{cases}$ Note: X and Y each have the same distribution as the absolute value of a $N(0, 0.5)$ random variable.

(b) $f_{X,Y}(u,v) = \begin{cases} -\frac{ln(u)v^2}{21} & 0 \leq u \leq 1, 1 \leq v \leq 4 \\ 0 & \text{else.} \end{cases}$

Solution: Yes. $f_X(u) = \begin{cases} -\ln(u) & u \geq 0 \\ 0 & \text{else} \end{cases}$ and $f_Y(v) = \begin{cases} \frac{v^2}{21} & 1 \leq v \leq 4 \\ 0 & \text{else.} \end{cases}$

(c) $f_{X,Y}(u,v) = \begin{cases} \frac{(96)u^2v^2}{\pi} & u^2+v^2 \le 1 \\ 0 & \text{else.} \end{cases}$

Solution: No. The support set of $f_{X,Y}$ is the unit disk, $\{(u,v) : u^2+v^2 \le 1\}$, which is not a product set. For example, $(0, 0.8)$ and $(0.8, 0)$ are in the support of $f_{X,Y}$ but $(0.8, 0.8)$ is not.

5. **[Joint densities, PS 11, number 1]**
X and Y are two random variables with the following joint pdf:

$$f_{X,Y}(u,v) = \begin{cases} A(1-|u-v|) & 0<u<1, 0<v<1 \\ 0 & else \end{cases}$$

(a) Find A.
Solution:

$$f_{X,Y}(u,v) = \begin{cases} A(1-(u-v)), & u \ge v \\ A(1+(u-v)), & u < v \end{cases}$$

$$\int_0^1 \int_0^u A(1-u+v)dvdu + \int_0^1 \int_0^v A(1+u-v)dudv = 1 \to A = 3/2$$

(b) Find marginal pdfs for X and Y.
Solution: The support of f_X is the interval $[0,1]$. For $0 \le u \le 1$,

$$f_X(u) = \int f(u,v)dv = \int_0^u A(1-u+v)dv + \int_0^1 A((1+u-v)dv$$

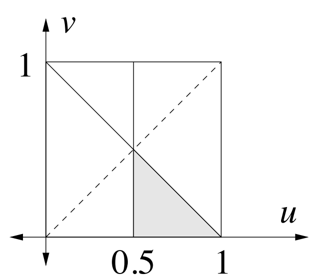
Therefore,

$$f_X(u) = \begin{cases} \frac{-3u^2}{2} + \frac{3u}{2} + \frac{3}{4} & 0<u<1 \\ 0 & \text{else.} \end{cases}$$

Since $f_{X,Y}(u,v) = f_{X,Y}(v,u)$, or in other words, (X,Y) has the same pdf as (Y,X), the pdfs f_Y and f_X are the same.

(c) Find $P\{X > Y\}$.
Solution: By symmetry, $P\{X > Y\} = 1/2$.

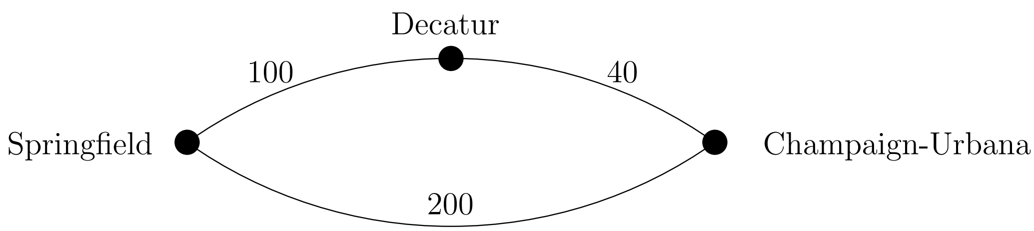
(d) Find $P(X+Y<1|X>1/2)$.
Solution: First, use the pdf of X to compute $P\{X \ge \frac{1}{2}\} = \int_{0.5}^1 f_X(u)du = 0.5$. Also, $P\{X+Y<1, X>1/2\}$ is the integral of the joint pdf over the shaded region:



So $P\{X+Y<1, X>1/2\} = \int_0^1 \int_{0.5}^{1-u} \frac{3}{2}(1-u+v)dvdu = \frac{3}{32}$. Therefore,

$$P(X+Y<1|X>1/2) = \frac{P\{X+Y<1, X>1/2\}}{P\{X>1/2\}} = 3/16.$$

6. **[15 points]** (Fall 2009, Exam 2, number 1) Messages can be transmitted between Champaign-Urbana and Springfield either directly or via Decatur on communication links whose capacities are indicated on the diagram shown below. Assume that the three links fail independently with equal probability $\frac{1}{2}$. Let X denote the communication capacity between Champaign-Urbana and Springfield.



(a) **[5 points]** What values can X take on?
Solution: It is easy to get that X can take on values 0, 40, 200, and 240.

(b) **[10 points]** Find the expected value of X.
Solution: X has value 240 when all links are working, which occurs with probability $\frac{1}{8}$.
X has value 40 when the direct link between CU and Springfield has failed while the other two links are working, which also occurs with probability $\frac{1}{8}$.
X has value 200 when the direct link between CU and Springfield is working while *at least one* of the other two links has failed, which occurs with probability $\frac{3}{8}$. Hence,

$$\mathsf{E}[X] = 240 \times \frac{1}{8} + 200 \times \frac{3}{8} + 40 \times \frac{1}{8} = \frac{240 + 3 \times 200 + 40}{8} = \frac{880}{8} = 110.$$

7. **[Fall 2008 Exam 2, number 3]**
Let X denote a Gaussian random variable with mean 10 and variance 100. Find the value of $P(X > 71.7|X > 29.6)$. A table of $\Phi(x)$, the CDF of the standard Gaussian distribution, is attached to this examination.

Solution:

$$\begin{aligned}
P\{X > 31.7, X > 29.6\} &= P\{X > 31.7\} = P\left\{\frac{X-10}{10} > 2.17\right\} = 1 - \Phi(2.17) = 1 - 0.9850 = 0.150 \\
P\{X > 29.6\} &= P\left\{\frac{X-10}{10} > 1.96\right\} = 1 - \Phi(1.96) = 1 - 0.9750 = 0.0250 \\
P(X > 31.7|X > 29.6) &= \frac{P\{X > 31.7, X > 29.6\}}{P\{X > 29.6\}} = \frac{0.150}{0.250} = 0.6
\end{aligned}$$

8. **[Spring 2008, Exam II, problem 2]**

(a) W denotes a *uniform* random variable with mean 1 and variance 3. Find $P\{W < 0\}$.
Solution: If $W \sim U[a,b]$, then $E[W] = (a+b)/2 = 1$ while $\mathrm{Var}(W) = (b-a)^2/12 = 3$ giving that $(b-a)^2 = 36$, i.e., $b - a = 6$ from which we get that $a = -2, b = 4$. Hence, $P\{W < 0\} = 1/3$.

(b) Suppose X is an exponential random variable with parameter λ. Calculate the pdf of the random variable $Y = \sqrt{2\lambda X}$.
Solution: Since X is non-negative, so is Y. So for any $v \geq 0$, we have:
$F_Y(v) = P(Y \leq v) = P\sqrt{2X\lambda} \leq v = PX \leq \frac{v^2}{2\lambda} = F_X\left(\frac{v^2}{2\lambda}\right) = 1 - \exp\left(-\frac{v^2}{2}\right)$
$\Rightarrow f_Y(v) = v\exp\left(-\frac{v^2}{2}\right)$ Thus for all v we have that:
$f_Y(v) = \begin{cases} v\exp\left(-\frac{v^2}{2}\right) & v \geq 0 \\ 0, & \text{otherwise.} \end{cases}$ This is called a *Rayleigh density function.*

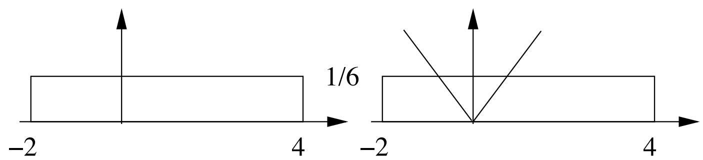
(c) Suppose that Z is a standard Gaussian random variable. Find $E[|Z|]$.

Solution: By LOTUS, $\mathsf{E}[|\mathbb{Z}|] = \displaystyle\int_{-\infty}^{\infty} |v| \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{v^2}{2}\right) dv = 2\int_0^{\infty} v \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{v^2}{2}\right) dv = \sqrt{\frac{2}{\pi}}$ since $v\exp(-v^2/2), v \geq 0$ is a valid pdf (cf. part (b) above!)

9. **[Fall 2009 Exam 2, number 5 points]** X denotes a *uniform* random variable with mean 1 and variance 3.

   (a) **[10 points]** Find $P\{X < 0\}$.

   Solution: A random variable uniformly distributed on $[a, b]$ has mean $\dfrac{a+b}{2}$ and variance $\dfrac{(b-a)^2}{12}$. Hence, we have that $b - a = 6$, and $b + a = 2$, giving $a = -2, b = 4$. The pdf is thus as shown in the left-hand figure below.



   By inspection, $P\{X < 0\} = \frac{1}{3}$

   (b) **[15 points]** Find the pdf of $Y = |X|$. In order to receive full credit, you must specify the value of $f_Y(v)$ for all $v, -\infty < v < \infty$.

   Solution: $Y = |X|$ takes on values in $[0, 4]$, and hence $F_Y(v) = 0$ for $v < 0$ and $F_Y(v) = 1$ for $v > 4$.
   For any $v, 0 \leq v \leq 2, F_Y(v) = P\{Y \leq v\} = P\{[-v \leq X \leq v\} = v/3$.
   For any $v, 2 \leq v \leq 4, F_Y(v) = P\{Y \leq v\} = P\{X \leq v\} = (v + 2)/6$.
   Hence, $f_Y(v) = \dfrac{d}{du}F_Y(v) = \begin{cases} \frac{1}{3}, & 0 \leq v \leq 2, \\ \frac{1}{6}, & 2 < v \leq 4, \\ 0, & \text{elsewhere.} \end{cases}$
   It is easy to verify that this is a valid pdf, and thus we have not made any obvious errors.