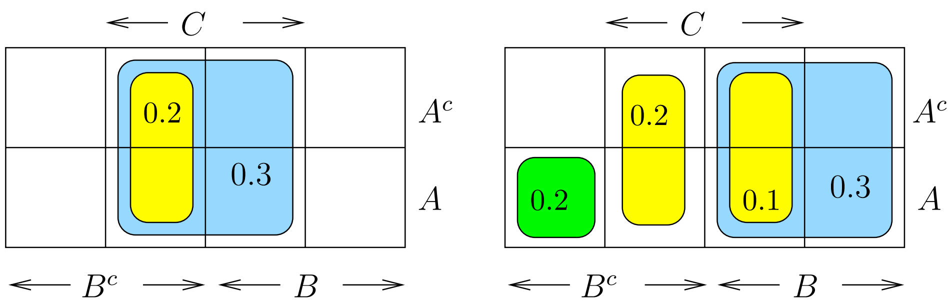
University of Illinois Fall 2009

ECE 313: Hour Exam I

Monday October 12, 2009
7:00 p.m. — 8:00 p.m.
100 Noyes Laboratory

1. **[15 points]** Let A, B, and C denote three events defined on a sample space Ω, and suppose that $P(A) = 0.6, P(B) = P(C) = 0.3$, and $P(B^c \cap C) = P(A \cap B^c \cap C^c) = 0.2$.

   Solution: The Karnaugh maps shown below are very useful in visualizing the problem.



   (a) **[5 points]** Find $P(B \cap C)$.
   Solution: $P(B \cap C) = P(C) - P(B^c \cap C) = 0.3 - 0.2 = 0.1$.
   Note that $P(B \cup C) = P(B) + P(B^c \cap C) = 0.5$.

   (b) **[5 points]** Find $P(B \cap C^c)$.
   Solution: $P(B \cap C^c) = P(B) - P(B \cap C) = 0.3 - 0.1 = 0.2$.

   (c) **[5 points]** Find $P((A \cup B \cup C)^c)$.
   Solution: $P((A \cup B \cup C)^c) = 1 - P(A \cup B \cup C) = 1 - [P(B \cup C) + P(A \cap B^c \cap C^c)]$ $= 1 - [0.5 + 0.2] = 0.3$. Note that $P((A \cup B \cup C)^c) = P(A^c \cap B^c \cap C^c)$, and also that $P(B^c \cap C^c) = 1 - P(B \cup C) = 0.5$.

2. **[10 points]** A and B are events defined on a sample space Ω. Assume $P(A), P(B) > 0$. Mark each of the two statements below as TRUE or FALSE. *No justification is needed.*

TRUE	FALSE	
□	□	If $P(A \mid B) = P(B \mid A)$, then $P(A) = P(B)$.
□	□	$P(A \mid B)P(B) + P(A^c \mid B)P(B) = P(B)$.

   Solution: The statement "If $P(A \mid B) = P(B \mid A)$, then $P(A) = P(B)$" seems to be true at first glance, but is in fact FALSE in general. (The statement is true if we are also told that $P(A \cap B) > 0$.)

   The statement "$P(A \mid B)P(B) + P(A^c \mid B)P(B) = P(B)$" is TRUE. The left side is equal to $P(A \cap B) + P(A^c \cap B)$ which is the same as $P(B)$ by the third axiom.

3. **[30 points]** *Especially in this problem, you must provide sufficient explanation to justify your numerical answers.*

A fair coin is tossed repeatedly until a Head occurs. N denotes the number of tosses.

Solution: Clearly, N is a *geometric* random variable with parameter $p = \frac{1}{2}$.

(a) **[5 points]** What is the expected value of N?

Solution: Since N is a *geometric* random variable with parameter $p = \frac{1}{2}$, $\mathsf{E}[N] = \frac{1}{p} = 2$ using a formula that you might have on your sheet of notes.
Alternatively, $P\{N = k\} = 2^{-k}$ for $k \geq 1$, and hence
$\mathsf{E}[N] = 1\cdot 2^{-1} + 2\cdot 2^{-2} + 3\cdot 2^{-3} + \cdots = \frac{1}{2}\cdot\left[1 + 2\cdot 2^{-1} + 3\cdot 2^{-2} + \cdots\right] = \frac{1}{2}\cdot\frac{1}{(1-\frac{1}{2})^2} = 2$ where the series sum was found on Problem Set 0, and its importance in ECE 313 noted.

(b) **[5 points]** Find the *numerical value* of $P\{N > 5\}$.

Solution: Since N is a *geometric* random variable with parameter $p = \frac{1}{2}$, $P\{N > 5\} = (1-p)^5 = \frac{1}{32}$ using a formula that you might have on your sheet of notes.
Alternatively, $N > 5$ if and only if the first five tosses resulted in Tails, and this has probability $\left(\frac{1}{2}\right)^5 = \frac{1}{32}$.
Doing it an even harder way,
$P\{N > 5\} = 2^{-6} + 2^{-7} + 2^{-8} + \cdots = 2^{-6}[1 + 2^{-1} + 2^{-2} + \cdots] = 2^{-6}\cdot\frac{1}{1-\frac{1}{2}} = 2^{-5}$. Yowza!

(c) **[10 points]** Given that the event $P\{N > 5\}$ occurred, what is the expected value of N?

Solution: N is the waiting time for a Head to occur, and $\mathsf{E}[N]$ is the average waiting time for a Head to occur. Given that the event $P\{N > 5\}$ occurred, that is, the first five tosses resulted in Tails, the waiting time for a Head (beginning with the 6th toss) is still the same by the *memoryless property* of the geometric distribution. Hence, $\mathsf{E}[N \mid N > 5] = 5 + 2 = 7$ using the result from part (a).
Alternatively, *conditioned on Tails on the first five tosses*,
$P\{N = 6\} = 2^{-1}, P\{N = 7\} = 2^{-2}, P\{N = 8\} = 2^{-3}$ and so on. Hence, the average value of N *conditioned on Tails on the first five tosses* is $6\cdot 2^{-1} + 7\cdot 2^{-2} + 8\cdot 2^{-3} + \cdots$
$= (5+1)\cdot 2^{-1} + (5+2)\cdot 2^{-2} + (5+3)\cdot 2^{-3} + \cdots$
$= 5\cdot\left[2^{-1} + 2^{-2} + 2^{-3} + \cdots\right] + 1\cdot 2^{-1} + 2\cdot 2^{-2} + 3\cdot 2^{-3} + \cdots = 5 + 2 = 7$ using the sum evaluated in part (a).

(d) **[10 points]** Find the *numerical value* of $\mathsf{E}[\cos(\pi N)]$.

Solution: $\cos(\pi N) = \begin{cases} -1 & \text{if } N \text{ is odd,} \\ +1 & \text{if } N \text{ is even,} \end{cases} = (-1)^N$, and hence

$$\mathsf{E}[\cos(\pi N)] = -\frac{1}{2} + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^3 + \cdots = \left(-\frac{1}{2}\right) + \left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^3 + \cdots$$

$= \dfrac{1}{1 - (-\frac{1}{2})} - 1 = -\dfrac{1}{3} = P(\text{Wilma wins}) - P(\text{Fred wins})$ when they play with a fair coin.

4. **[20 points]** A fair coin is tossed 10 times.

Calculate the probability that the first 5 tosses are all Tails given that a total of 8 Tails occurred on the 10 tosses.

Solution: Let A denote the event that the first 5 tosses result in Tails, and B the event that 8 Tails occurred on the 10 tosses. We are asked to find $P(A \mid B) = P(A \cap B)/P(B)$. Now, the number of Tails on 10 tosses is a *binomial* random variable N with parameters $(10, \frac{1}{2})$, and thus, we have

$$P(B) = P\{N = 8\} = \binom{10}{8}\left(\frac{1}{2}\right)^8\left(1 - \frac{1}{2}\right)^{10-8} = \binom{10}{2}\left(\frac{1}{2}\right)^{10} = \frac{10\times 9}{1\times 2}\times 2^{-10}.$$

On the other hand, $A \cap B = A \cap C$ where C is the event that 3 Tails occurred on the *last* 5 tosses. Since what occurred on the last five tosses is independent of what occurred on the first five tosses, we have that

$$P(A \cap B) = P(A \cap C) = 2^{-5}\binom{5}{3}\left(\frac{1}{2}\right)^{3}\left(\frac{1}{2}\right)^{5-3} = \binom{5}{2}\left(\frac{1}{2}\right)^{10} = \frac{5 \times 4}{1 \times 2} \times 2^{-10}.$$

Hence, $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{5 \times 4}{10 \times 9} = \dfrac{2}{9}$.

5. **[25 points]** Dilbert has 3 coins in his pocket, 2 of which are fair coins while the third is a biased coin with $P(H) = p \neq \frac{1}{2}$. The probability that a coin chosen at random from his pocket will land Tails is $\frac{7}{12}$.

   (a) **[10 points]** What is the value of p?
   Solution: $P(T) = P(T \mid \text{fair coin})P(\text{fair coin}) + P(T \mid \text{biased coin})P(\text{biased coin})$
   $= \dfrac{1}{2} \cdot \dfrac{2}{3} + (1-p) \cdot \dfrac{1}{3} = \dfrac{7}{12} \Rightarrow p = \dfrac{1}{4}$.

   (b) **[15 points]** Dilbert picks two coins at random from his pocket, tosses each coin once, and observes a Head and a Tail. What is the conditional probability that both coins are fair?

   Solution: $P(\text{one Head, one Tail} \mid \text{two fair coins}) = P(\{HT, TH\}) = \dfrac{1}{2} \cdot \dfrac{1}{2} + \dfrac{1}{2} \cdot \dfrac{1}{2} = \dfrac{1}{2}$.
   $P(\text{one Head, one Tail} \mid \text{one fair, one biased})$
   $= P(\{\text{fair} = H, \text{biased} = T\}) + P(\{\text{biased} = H, \text{fair} = T\}) = \dfrac{1}{2} \cdot (1-p) + p \cdot \dfrac{1}{2} = \dfrac{1}{2}$
   regardless of the value of p. Therefore, the theorem of total probability gives

$$\begin{aligned} P(\text{one Head, one Tail}) &= P(H, T \mid \text{two fair coins})P(\text{two fair coins}) \\ &\quad + P(H, T \mid \text{one fair, one biased})P(\text{one fair, one biased}) \\ &= \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{6} + \frac{1}{3} = \frac{1}{2}, \end{aligned}$$

   and the *conditional* probability of both coins being fair given that one Head and one Tail was observed is, by Bayes' formula,

$$P(\text{both coins fair}|\text{one Head, one Tail}) = \frac{P(H, T|\text{two fair coins})P(\text{two fair coins})}{P(\text{one Head, one Tail})} = \frac{1/6}{1/2} = \frac{1}{3}$$

   the same as the unconditional probability! What does this tell you about the events $\{\text{both coins fair}\}$ and $\{\text{one Head, one Tail}\}$?