

ECE 313: Problem Set 7
Independent Events, System Reliability

This Problem Set contains eight problems

Due: Wednesday, October 22nd at 4 p.m.
Reading: Ross Ch. 3; PowerPoint Lectures 18-20
Noncredit Exercises: **Chapter 3:** Problems 53, 58, 59, 62, 63, 66, 70-74, 78, 81
Theoretical Exercises 6, 9, 11 25, 28; Self-Test Problems 11-21

1. An ECE 313 student seeks predictions from three psychics A, B, and C as to whether the student will pass the course. The psychics predict that the student will pass with probabilities $P(A) = \frac{1}{3}$, $P(B) = \frac{1}{2}$, and $P(C) = \frac{3}{4}$ respectively. Assume that A, B, and C are *mutually independent* events, and let D denote the event that at least two of the three psychics predict that the student will pass.
 - (a) What is $P(D)$?
 - (b) Given that event D occurred, what is the probability that the most pessimistic psychic A predicted that the student will *fail* ECE 313?
2. An experiment consists of three independent tosses of a coin. Let A and B respectively denote the events that the first and third tosses result in a Head, and let C denote the event that exactly two Heads occur *and* they occur on two successive tosses. Suppose that the coin is fair.
 - (a) Are A and B physically independent events? Explain.
 - (b) Are A and C stochastically independent events? Are they physically independent events? That is, are A and C independent if the coin were a biased coin?
 - (c) Are B and C stochastically independent events? Are they physically independent events? That is, are B and C independent if the coin were a biased coin?
 - (d) Are A, B, and C independent events?
 - (e) Are A, B, and C *pairwise* independent events?
3. There are 13 different pairs of shoes in a barrel. Suppose that two shoes are picked at random and one at a time *without replacement* from the barrel.
 - (a) What is the probability that the first shoe is a left shoe and the second shoe is a right shoe? What is the probability that the first shoe is a right shoe and the second shoe is a left shoe? What is the probability of getting two left shoes? What is the probability of getting two right shoes? Why do these probabilities not add up to 1?
 - (b) Two *more* shoes are picked at random without replacement from the barrel. Note that the first two shoes picked *are* not put back in the barrel before the two additional shoes are drawn. Use the chain rule of conditional probability and an analysis similar to the birthday surprise problem to find the probability that there are *no matching pairs* of shoes in the four shoes picked from the barrel.
 - (c) Given that there are no matching pairs of shoes in the four that have been picked, what is the (conditional) probability that all four shoes are left shoes?

4. Each box of Cornies, the breakfast of silver medalists, contains a picture of either Britney Spears or Paris Hilton. The purchase of each box of Cornies can be regarded as an independent trial of an experiment on which events S and H occur with probabilities $\frac{1}{4}$ and $\frac{3}{4}$ respectively. Let $\mathcal{Y}$ denote the number of boxes of Cornies purchased till the experimenter has acquired at least two pictures of each woman.

- (a) For $k \geq 4$, what is $P\{\mathcal{Y} = k\}$?
 (b) Find $E[\mathcal{Y}]$.

5. [“From the town of Bedrock, They’re a page right out of history ...”]
 Fred and Wilma play a series of games by *taking turns* tossing a fair coin with $P(\text{Heads}) = p$ and $P(\text{Tails}) = q = 1 - p$ until one of them tosses a Head and wins the game. Fred (that MCP!) tosses first in the first game. Note that if Fred tosses a Head on the first try, Wilma loses the game without getting to toss the coin at all! The *loser* of a game gets to toss first in the *next* game. Let F_n and W_n respectively denote the events that Fred and Wilma won the n -th game.

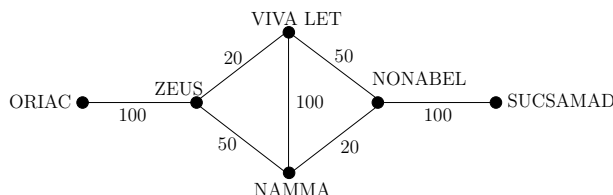
- (a) Show that $P(F_1) = \frac{2}{3}, P(W_1) = \frac{1}{3}, P(F_2) = \frac{4}{9}, P(W_2) = \frac{5}{9}$.
 (b) Express $P(F_{n+1})$ and $P(W_{n+1})$ in terms of $P(F_n)$ and $P(W_n)$.
 (c) Your answers to part (b) are a pair of *difference* equations for $P(F_n)$ and $P(W_n)$. Show that the solutions are

$$P(F_n) = \frac{1}{2} \left[1 - \left(\frac{-1}{3} \right)^n \right] \quad \text{and} \quad P(W_n) = \frac{1}{2} \left[1 + \left(\frac{-1}{3} \right)^n \right]$$

[Hint If you never learned how to solve difference equations in Math 386 or ECE 410, assume that for *each integer* $n \geq 1$, it is possible to express $P(F_n)$ as $a + b\alpha^n$. Substitute into the difference equation and use the fact that equality must hold for *all* n to find a and α . The value of b is obtained from the initial condition $P(F_1) = \frac{2}{3}$. The solution for $P(W_n)$ is similar but the initial condition is $P(W_1) = \frac{1}{3}$.]

- (d) Show that $\lim_{n \rightarrow \infty} P(F_n) = \lim_{n \rightarrow \infty} P(W_n) = \frac{1}{2}$ so that if Fred and Wilma are still playing, the game is quite close to being fair by now!
6. The Senate of a certain country has 100 members consisting of 43 Conservative Republicans, 21 Conservative Democrats, 12 Liberal Republicans, and 24 Liberal Democrats. Before each vote, the groups caucus separately. Each group decides independently of the other groups whether to support or oppose the motion. All members of the group then vote in accordance with the caucus decision. If you believe that this is the way politics works, I have this beautiful skyscraper on Wacker Drive in Chicago that I am willing to sell to you at a real bargain price ...
- (a) Let A, B, C , and D respectively denote the events that the four groups vote to eliminate all income taxes on capital gains. Suppose that these independent events have probabilities $P(A) = 0.9, P(B) = 0.6, P(C) = 0.5$ and $P(D) = 0.2$. What is the probability that the bill passes?

- (b) The President vetoes the bill as a budget-breaker. Let E, F, G , and H respectively denote the independent events that the four groups support the motion to override the veto. If $P(E) = 0.99, P(F) = 0.4, P(G) = 0.6$, and $P(H) = 0.1$, what is the probability that the motion to override the veto passes ?
- Political innocents are reminded that a simple majority (51 or more votes) is required to pass a bill, and a two-thirds majority (67 or more votes) to override a veto.
7. MiddleEast Bell, a wholly owned subsidiary of Psingular Corp. has built a telephone network as shown below. Each link fails with probability p and link failures are



- independent events. If a link fails, switches automatically re-route calls so as to avoid the failed link (if possible).
- What is the probability of being able to call from ORIAC to SUCSAMAD?
 - Given that it is possible to call from ORIAC to SUCSAMAD, what is the conditional probability that the ZEUS to NAMMA link is in working condition?
 - The link capacities (i.e., the numbers of telephone calls that the links can carry (in either direction)) are as marked on the diagram. Let $\mathcal{X}$ denote the number of calls that can be made from ORIAC to SUCSAMAD. Find the pmf and the expected value of $\mathcal{X}$.
8. The ToyAuto Company needs to decide which of the following two methods provides more reliable transportation:
- a single gigantic car with N engines, N transmissions, N brakes, ... etc. that works (i.e. provides us with transportation) as long as *at least one* of its engines and *at least one* of its transmissions, and *at least one* of its brakes ... works.
 - N separate ordinary cars that fail as soon as any one of their parts fail, but which together provide us with transportation as long as at least one car is in working condition.

Each car is made of M different types of parts, and (at least) one part of each different type must work for the car to work. Each part fails with probability p and all the failures are independent events.

- For each method, find the probability of system failure (we have no transportation!) in terms of p, N and M .
- Suppose that $M = 5$ and $p = 0.2$. If it is desired that the system failure probability be less than 0.001, what should N be with each method?
- Repeat part (b) assuming that $M = 1000$.