

ECE 313: Problem Set 3

Axioms of Probability; Random Variables

This Problem Set contains seven problems

Due: Wednesday September 17, 4 p.m.
Reading: Ross, Chapter 2, Sections 1-5, and Chapter 4
Noncredit Exercises: DO NOT turn these in.

Chapter 2: Problems 3, 4, 9, 10, 11, 12, 14;
 Theoretical Exercises 1-3, 6, 10, 11, 16, 19, 20; Self-Test Problems 1-8
Chapter 4: Problems 1, 3-9, 13

1. For $\alpha > 0$, the *Gamma function* $\Gamma(\alpha)$ is defined as $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} \exp(-t) dt$.
 - (a) Show by integration that $\Gamma(1) = 1$.
 - (b) Use integration by parts to show that $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$, and use this to deduce that $\Gamma(n + 1) = n!$.
 - (c) If α is not an integer, show that $\Gamma(\alpha + 1) = \alpha(\alpha - 1)(\alpha - 2) \cdots \Gamma(\alpha - \lfloor \alpha \rfloor)$ where $\lfloor \alpha \rfloor$ is the *integer part* of α . Note that $0 < \alpha - \lfloor \alpha \rfloor < 1$. For $0 < \beta < 1$, $\Gamma(\beta)$ must be evaluated by numerical integration, except...
 - (d) show that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. Hint: make the change of variables $\sqrt{2t} = x$. Then show that

$$\Gamma\left(\frac{1}{2}\right) = \left(2 \int_0^\infty \int_0^\infty \exp(-[x^2 + y^2]/2) dx dy\right)^{1/2}$$

where the double integral can be evaluated by a change to polar coordinates as in Problem 5(b) of Problem Set 1.

2. The binomial coefficient $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ denotes the number of subsets of size k drawn from a set of size n .
 - (a) Give a combinatorial proof that $\binom{m+n}{k} = \sum_{i=0}^k \binom{m}{i} \binom{n}{k-i}$. Hint: How many committees of size k can be formed from a group of m men and n women?
 - (b) What is the coefficient of x^k in the polynomial $(1+x)^{m+n}$?
 Now write $(1+x)^{m+n} = (1+x)^m(1+x)^n$ and find the coefficient of x^k on the right hand side in terms of the coefficients of $(1+x)^m$ and $(1+x)^n$. If you have taken ECE 410, you might recognize a discrete convolution here
 - (c) Use the result of part (a) to show that $\binom{2n}{n} = \sum_{i=0}^n \left[\binom{n}{i}\right]^2$.
3. Ross, Chapter 2, Problem 13 (page 57).

4. An experiment consists of observing the contents of an 8-bit register. We assume that all 256 byte values are equally likely to be observed.
 - (a) Let A denote the event that the least significant bit is a ONE. What is $P(A)$?
 - (b) Let B denote the event that the register contains 5 ONES and 3 ZEROes. What is $P(B)$?
 - (c) What is $P(A \cup B)$? What is $P(A \cap B)$? What is the probability that exactly one of the two events A and B occurs, i.e. what is $P(A \oplus B)$?
5. [“Eat your broccoli, dear; it’s good for you”]
 - (a) Your mother has bought three servings of broccoli and two servings of cauliflower for next week (Monday through Friday) and will serve one vegetable each day.
 - i. Define an appropriate sample space that includes all possible outcomes of this experiment (which you don’t get to perform: it is performed on you!). Assume that all the outcomes are equally likely.
 - ii. What is the probability of having broccoli on Monday?
 - iii. What is the probability of having broccoli on Monday and Friday?
 - iv. What is the probability of having broccoli on Monday, Wednesday, and Friday?
 - (b) Suppose instead that A , B , and C denote the events that your mother serves asparagus, broccoli or cauliflower for dinner. From (bitter?) experience, you know that these events are mutually exclusive and that $P(A) = 0.2$, $P(B) = 0.5$, and $P(C) = 0.3$. Each day is an *independent trial*, that is, your mother, a lady of formidable temperament albeit limited culinary skills, makes *independent decisions* as to which vegetable to serve without taking into account your opinion that Cheetos is a vegetable that goes well with any entree. Over a three day period, what is the probability that
 - i. she serves the same vegetable on all three days?
 - ii. she serves the same vegetable exactly two days out of three?
 - iii. she serves different vegetables on the three days?
 - (c) **Optional for 0 Comp II credit:** Write a 500-word essay on why you like dorm food so much.
6. (a) $\Omega = \{0, 1, 2, \dots\}$ is a countably infinite sample space with $P(n) = \frac{(\ln 2)^n}{2(n!)}$ for all $n \geq 0$. Remember that $0! = 1$.
 - i. Show that $P(\Omega) = 1$ for this probability assignment.
 - ii. Prove that the probability that the outcome is an even number is $5/8$. Remember that 0 is an even number.
- (b) Bob & Carol & Ted & Alice take turns (in that order) tossing a coin with $P(H) = p$, $0 < p < 1$. The first one to toss a Head wins the game. Calculate their *win probabilities* $P(B)$, $P(C)$, $P(T)$, and $P(A)$ and show that
 - i. $P(B) > P(C) > P(T) > P(A)$.
 - ii. $P(B) + P(C) + P(T) + P(A) = 1$.
7. Ross, Chapter 4, Problem 2, (page 187).