

ECE 313: Problem Set 1

Calculus Tune-up

Due: Wednesday September 3 at 4 p.m.

Reading: Ross, Chapters 1 and 2

This Problem Set contains six problems

Note: Most of the topics covered on this problem set will be needed in ECE 313 after the drop deadline. Thus this problem set (which is based entirely on material covered in the prerequisites to ECE 313) is intended as a review and a diagnostic aid to help you identify topics that might need reviewing before embarking on ECE 313.

1. (a) Prove that $1 + x + x^2 + \cdots + x^{n-1} = \frac{1 - x^n}{1 - x}$ for all $x \neq 1$ and all integers $n \geq 1$.
 (b) Continuing to assume that n is a positive integer, what is the value of the sum $1 + x + x^2 + \cdots + x^{n-1}$ when $x = 1$? Does your answer equal $\lim_{x \rightarrow 1} \frac{1 - x^n}{1 - x}$?
 (c) Assuming that $|x| < 1$, find the sum of the series $1 + x + x^2 + \cdots$.
 Hint: it is the limit of the finite sum $1 + x + x^2 + \cdots + x^{n-1}$ as $n \rightarrow \infty$.
 (d) Find the derivative of $\exp(-x) \sum_{k=0}^n \frac{x^k}{k!}$. Simplify your answer as much as possible.
 What is the derivative of $\exp(-x) \sum_{k=0}^{\infty} \frac{x^k}{k!}$?
 (e) Find the maximum value of $x^{25}(1.00001)^{-x}$ in the interval $(0, \infty)$.
2. (a) For positive integers n and k , compute the k -th derivative of $f(x) = (1+x)^n$ using the chain rule, that is, *without* multiplying out the terms to get a polynomial in x (as in $(1+x)^2 = 1 + 2x + x^2$). Use these derivatives to find the MacLaurin series (Taylor series in the vicinity of 0) for $(1+x)^n$.
 (b) According to the textbook (Equation 4.2 in Chapter 1 with $y = 1$),

$$(1+x)^n = \sum_{k=0}^n \binom{n}{k} x^k \quad \text{where} \quad \binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Does your answer to part (a) match this result? If so, congratulations! You have just proved the binomial theorem for positive integer exponents.

- (c) Now consider the function $g(x) = (1-x)^{-n}$ where n is a positive integer. Does the MacLaurin series for $g(x)$ contain terms of degree $> n$? If so, what is the term for degree $n+1$? If not, what is the highest degree term?
- (d) Use the result of part (c) to write down the MacLaurin series for $(1-x)^{-1}$ and $(1-x)^{-2}$. These results together with the one of parts (a) and (b) will be needed so often in ECE 313 that it is recommended that you memorize them.
- (e) Find the MacLaurin series for $(1+x)^a$ where a is a real number and not necessarily an integer.

3. (a) Evaluate the integrals $\int_{-1}^2 |x| dx$ and $\int_{-2}^1 x(1-x)^{19} dx$.
- (b) Does there exist a *nonnegative* function $f(x)$ such that $\int_{-2}^1 f(x) dx < 0$. [Hint: Does either function of part (a) satisfy these conditions?]
- (c) Let $f(x)$ denote a differentiable function with derivative $g(x)$. Which of the following statements are true? In parts (v) and (vi), C denotes an arbitrary constant.
- (i) $\frac{d}{dx} f(-x) = -g(x)$ (ii) $\frac{d}{dx} f(x^2) = g(x^2)$
- (iii) $\frac{d}{dx} \exp(f(x)) = \exp(f(x))g(x)$ (iv) $\frac{d}{dx} \exp(f(x^2)) = \exp(f(x))g(x^2)$
- (v) $\int g(-x) dx = f(-x) + C$ (vi) $\int g(x^2) dx = f(x^2)/(2x) + C$
4. Let $I = \int_{-1}^1 \frac{2}{1+x^2} dx$ and $J = \int_{-1}^1 \frac{-2}{1+y^2} dy$, and note that $I = -J$.
- (a) Show that $I = \pi$. [Hint: the integrand is the derivative of $\arctan(x)$.]
- (b) Use the substitution $y = 1/x$ to transform I into J .
- (c) In part (b), you have shown that $I = J$. It has also been noted that $I = -J$. Now, $I = J$ and $I = -J$ both hold if and only if $I = J = 0$. Since you showed in part (a) that $I = \pi$, does this mean that $\pi = 0$?
5. (a) Find the integral of $f(x, y) = \max(x, y)$ over $\{(x, y) : 0 < x < 2, 0 < y < 1\}$.
- (b) Find the integral of $(x^2 + y^2)^{-2}$ over $\{(x, y) : x^2 + y^2 > 4\}$.
6. (a) What is the derivative of $\exp(-x^2/2)$?
- (b) Evaluate $\int_0^\infty x \exp(-x^2/2) dx$.
- (c) Evaluate $\int_{-1}^1 x^3 \exp(-x^2/2) dx$. (Hint: sketch the integrand first!)