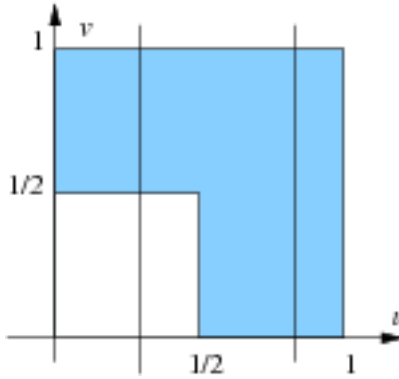


ECE 313: Solutions to Problem Set 13

1. $E[\mathcal{X}] = 1, E[\mathcal{Y}] = 4, \text{var}(\mathcal{X}) = 4, \text{var}(\mathcal{Y}) = 9$, and $\rho_{\mathcal{X}, \mathcal{Y}} = 0.1$.
 - (a) $E[\mathcal{Z}] = E[2(\mathcal{X} + \mathcal{Y})(\mathcal{X} - \mathcal{Y})] = 2E[\mathcal{X}^2 - \mathcal{Y}^2] = 2E[\mathcal{X}^2] - 2E[\mathcal{Y}^2] = 2[4 + 1^2] - 2[9 + 4^2] = -40$.
 - (b) $\text{cov}(\mathcal{T}, \mathcal{U}) = \text{cov}(2\mathcal{X} + \mathcal{Y}, 2\mathcal{X} - \mathcal{Y}) = 4 \cdot \text{cov}(\mathcal{X}, \mathcal{X}) + 2 \cdot \text{cov}(\mathcal{Y}, \mathcal{X}) - 2 \cdot \text{cov}(\mathcal{X}, \mathcal{Y}) - \text{cov}(\mathcal{Y}, \mathcal{Y})$
 $= 4 \cdot \text{var}(\mathcal{X}) + 2 \cdot \text{cov}(\mathcal{X}, \mathcal{Y}) - 2 \cdot \text{cov}(\mathcal{X}, \mathcal{Y}) - \text{var}(\mathcal{Y}) = 4 \cdot \text{var}(\mathcal{X}) - \text{var}(\mathcal{Y}) = 4 \cdot 4 - 9 = 7$.
 - (c) $E[\mathcal{W}] = E[3\mathcal{X} + \mathcal{Y} + 2] = 3E[\mathcal{X}] + E[\mathcal{Y}] + 2 = 9$.
 $\text{var}(\mathcal{W}) = \text{var}(3\mathcal{X} + \mathcal{Y} + 2) = 3^2 \cdot \text{var}(\mathcal{X}) + \text{var}(\mathcal{Y}) + 2 \cdot 3 \cdot 1 \cdot \text{cov}(\mathcal{X}, \mathcal{Y}) = 9 \cdot 4 + 9 + 6 \cdot 2 \cdot 3 \cdot 0.1 = 48.6$.
 - (d) $P\{\mathcal{W} > 0\} = 1 - \Phi\left(\frac{0 - 9}{\sqrt{48.6}}\right) = 1 - \Phi\left(-\frac{9}{\sqrt{48.6}}\right) = \Phi\left(\frac{9}{\sqrt{48.6}}\right)$.
2.
 - (a) $\text{var}(\mathcal{X} + \mathcal{Y}) = \text{var}(\mathcal{X}) + \text{var}(\mathcal{Y}) + 2 \cdot \text{cov}(\mathcal{X}, \mathcal{Y}) = 36$.
 $\text{var}(\mathcal{X} - \mathcal{Y}) = \text{var}(\mathcal{X}) + \text{var}(\mathcal{Y}) - 2 \cdot \text{cov}(\mathcal{X}, \mathcal{Y}) = 64$. Hence, $\text{cov}(\mathcal{X}, \mathcal{Y}) = -7$.
 From the above, $2 \cdot \text{var}(\mathcal{X}) + 2 \cdot \text{var}(\mathcal{Y}) = 8 \cdot \text{var}(\mathcal{Y}) = 100$, giving $\text{var}(\mathcal{Y}) = 12.5$,
 $\text{var}(\mathcal{X}) = 37.5$ and $\rho_{\mathcal{X}, \mathcal{Y}} = \text{cov}(\mathcal{X}, \mathcal{Y}) / \sqrt{\text{var}(\mathcal{X})\text{var}(\mathcal{Y})} = -7 / 12.5\sqrt{3}$.
 - (b) $\text{var}(\mathcal{X} + \mathcal{Y}) = \text{var}(\mathcal{X}) + \text{var}(\mathcal{Y}) + 2 \cdot \text{cov}(\mathcal{X}, \mathcal{Y}) = \text{var}(\mathcal{X} - \mathcal{Y}) = \text{var}(\mathcal{X}) + \text{var}(\mathcal{Y}) - 2 \cdot \text{cov}(\mathcal{X}, \mathcal{Y})$
 if and only if $\text{cov}(\mathcal{X}, \mathcal{Y}) = 0$, that is, if and only if $\mathcal{X}$ and $\mathcal{Y}$ are uncorrelated.
 - (c) No, whether $\text{var}(\mathcal{X})$ equals $\text{var}(\mathcal{Y})$ or not has no bearing on the question of whether $\text{cov}(\mathcal{X}, \mathcal{Y})$ is zero or not.
3. The random point $(\mathcal{X}, \mathcal{Y})$ is uniformly distributed on the shaded region shown in the figure below. Clearly, $f_{\mathcal{X}, \mathcal{Y}}(u, v) = \frac{4}{3}$ on this region.



- (a) $f_{\mathcal{X}}(u)$ is the area of the cross-section of the pdf surface along the line u . There are two cases to be considered, as shown in the left-hand figure above. It is obvious almost by

inspection that $f_{\mathcal{X}}(u) = \begin{cases} \frac{2}{3}, & 0 \leq u \leq \frac{1}{2}, \\ \frac{4}{3}, & \frac{1}{2} < u \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$

$$E[\mathcal{X}] = \int_{-\infty}^{\infty} u \cdot f_{\mathcal{X}}(u) du = \int_0^{1/2} u \cdot \frac{2}{3} du + \int_{1/2}^1 u \cdot \frac{4}{3} du = \frac{7}{12}$$

$$E[\mathcal{X}^2] = \int_{-\infty}^{\infty} u^2 \cdot f_{\mathcal{X}}(u) du = \int_0^{1/2} u^2 \cdot \frac{2}{3} du + \int_{1/2}^1 u^2 \cdot \frac{4}{3} du = \frac{5}{12}$$

$$\text{var}(\mathcal{X}) = E[\mathcal{X}^2] - (E[\mathcal{X}])^2 = \frac{5}{12} - \left(\frac{7}{12}\right)^2 = \frac{11}{144}$$

- (b) By symmetry, $f_{\mathcal{X}}$ and $f_{\mathcal{Y}}$ are the same function: $f_{\mathcal{Y}}(v) = \begin{cases} \frac{2}{3}, & 0 \leq v \leq \frac{1}{2}, \\ \frac{1}{3}, & \frac{1}{2} < v \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$

$$\mathbb{E}[\mathcal{Y}] = \frac{7}{12}, \quad \text{var}(\mathcal{Y}) = \frac{11}{144}$$

- (c) Given that $\mathcal{X} = \alpha$, the conditional pdf $f_{\mathcal{Y}|\mathcal{X}}(v|\alpha)$ is the cross-section of the pdf surface at $u = \alpha$ unitized to have area 1.

$$\text{For } 0 < \alpha < \frac{1}{2}: f_{\mathcal{Y}|\mathcal{X}}(v|\alpha) \sim \text{Uniform}[\frac{1}{2}, 1] \Rightarrow \mathbb{E}[\mathcal{Y}|\mathcal{X} = \alpha] = \frac{3}{4}, \quad \text{var}[\mathcal{Y}|\mathcal{X} = \alpha] = \frac{1}{48}$$

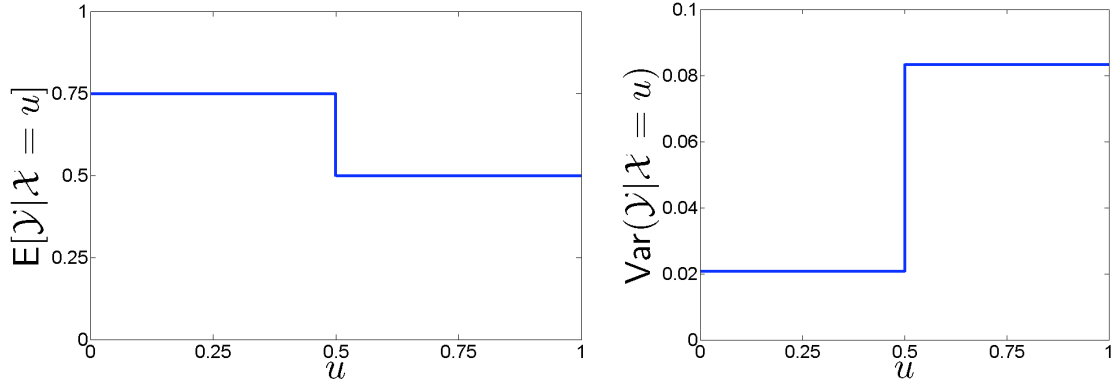
$$\text{For } \frac{1}{2} < \alpha < 1: f_{\mathcal{Y}|\mathcal{X}}(v|\alpha) \sim \text{Uniform}[0, 1] \Rightarrow \mathbb{E}[\mathcal{Y}|\mathcal{X} = \alpha] = \frac{1}{2}, \quad \text{var}[\mathcal{Y}|\mathcal{X} = \alpha] = \frac{1}{12}$$

$$(d) \quad 0 \leq v \leq \frac{1}{2}: f_{\mathcal{Y}}(v) = \int_0^1 f_{\mathcal{Y}|\mathcal{X}}(v|\alpha) f_{\mathcal{X}}(\alpha) d\alpha = \int_0^{1/2} (0) \cdot \frac{2}{3} d\alpha + \int_{1/2}^1 (1) \cdot \frac{4}{3} d\alpha = \frac{2}{3}$$

$$\frac{1}{2} \leq v \leq 1: f_{\mathcal{Y}}(v) = \int_0^1 f_{\mathcal{Y}|\mathcal{X}}(v|\alpha) f_{\mathcal{X}}(\alpha) d\alpha = \int_0^{1/2} (2) \cdot \frac{2}{3} d\alpha + \int_{1/2}^1 (1) \cdot \frac{4}{3} d\alpha = \frac{4}{3}$$

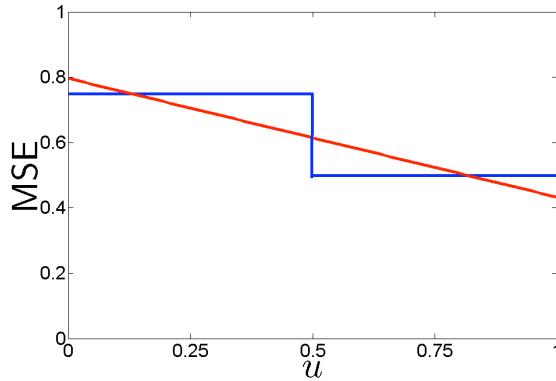
Yes, we get the same answer as in part (b).

- (e) See figures below.



$$(f) \quad \text{MSE of MMSE estimator} = \mathbb{E}[\text{var}(\mathcal{Y}|\mathcal{X} = u)] = \int_0^{1/2} \frac{1}{48} \cdot \frac{2}{3} du + \int_{1/2}^1 \frac{1}{12} \cdot \frac{4}{3} du = \frac{7}{432}$$

- (g) See figure below.



From the figure, $\hat{\mathcal{Y}}$ is larger than $\mathbb{E}[\mathcal{Y}|\mathcal{X} = 0]$, when $\mathcal{X} = u = 0$ and $\mathbb{E}[\mathcal{Y}|\mathcal{X} = 1]$ is larger than $\hat{\mathcal{Y}}$ when $\mathcal{X} = u = 1$.

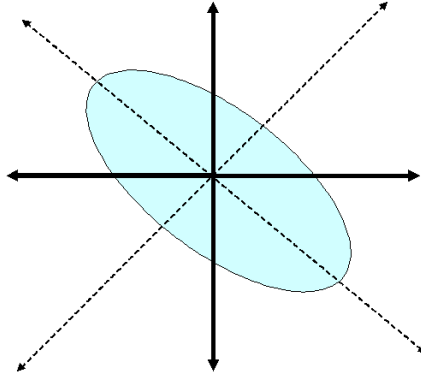
$$\text{var}(\mathcal{Y}) (1 - \rho_{\mathcal{X}, \mathcal{Y}}^2) = \frac{11}{144} \left(1 - \left(\frac{4}{11} \right)^2 \right) = \frac{35}{528} > \frac{7}{432} \text{ as it should be.}$$

4. Note that $E[\mathcal{X}] = E[\mathcal{Y}] = 0$ and hence $E[\mathcal{X}^2] = \sigma_1^2$ and $E[\mathcal{Y}^2] = \sigma_2^2$ while $\text{cov}[\mathcal{X}, \mathcal{Y}] = \rho \cdot \sigma_1 \sigma_2$.

(a)

$$\begin{aligned}
 E[\mathcal{Z}] &= E[\mathcal{X} \cos \theta + \mathcal{Y} \sin \theta] = E[\mathcal{X}] \cos \theta + E[\mathcal{Y}] \sin \theta = 0. \\
 E[\mathcal{W}] &= E[\mathcal{Y} \cos \theta - \mathcal{X} \sin \theta] = 0 \\
 \text{var}(\mathcal{Z}) &= E[\mathcal{Z}^2] - E[\mathcal{Z}]^2 = E[\mathcal{X}^2 \cos^2 \theta + \mathcal{Y}^2 \sin^2 \theta + 2\mathcal{X}\mathcal{Y} \sin \theta \cos \theta] \\
 &= \sigma_1^2 \cos^2 \theta + \sigma_2^2 \sin^2 \theta + 2\rho\sigma_1\sigma_2 \sin \theta \cos \theta \\
 \text{var}(\mathcal{W}) &= E[\mathcal{W}^2] - E[\mathcal{W}]^2 = E[\mathcal{X}^2 \sin^2 \theta + \mathcal{Y}^2 \cos^2 \theta - 2\mathcal{X}\mathcal{Y} \sin \theta \cos \theta] \\
 &= \sigma_1^2 \sin^2 \theta + \sigma_2^2 \cos^2 \theta - 2\rho\sigma_1\sigma_2 \sin \theta \cos \theta. \\
 \text{cov}(\mathcal{Z}, \mathcal{W}) &= E[\mathcal{Z}\mathcal{W}] - E[\mathcal{Z}]E[\mathcal{W}] = E[(\mathcal{X} \cos \theta + \mathcal{Y} \sin \theta)(\mathcal{Y} \cos \theta - \mathcal{X} \sin \theta)] \\
 &= E[\mathcal{Y}^2] \sin \theta \cos \theta - E[\mathcal{X}^2] \sin \theta \cos \theta + E[\mathcal{X}\mathcal{Y}](\cos^2 \theta - \sin^2 \theta) \\
 &= \sin \theta \cos \theta \cdot (\sigma_2^2 - \sigma_1^2) + (\cos^2 \theta - \sin^2 \theta) \cdot \rho\sigma_1\sigma_2 \\
 &= \frac{1}{2} \cdot \sin 2\theta \cdot (\sigma_2^2 - \sigma_1^2) + \cos 2\theta \cdot \rho\sigma_1\sigma_2
 \end{aligned}$$

- (b) $\mathcal{Z}$ and $\mathcal{W}$ are jointly Gaussian random variables and thus they are independent if $\text{cov}(\mathcal{Z}, \mathcal{W}) = 0$. We get independent random variables if we choose $\theta = \frac{1}{2} \arctan \left(\frac{2\rho \cdot \sigma_1 \sigma_2}{\sigma_1^2 - \sigma_2^2} \right)$.
 Note that if θ_0 is a solution to this equation, then $\theta_0 + \pi$ is also a solution, as are $\theta_0 \pm \pi/2$. That is, there are *four* different values of θ in the range $[0, 2\pi)$ that can be used to get independent Gaussian random variables from $\mathcal{X}$ and $\mathcal{Y}$. In particular, if $\sigma_1 = \sigma_2$, then θ can take on values $\pm\pi/4$ and $\pm 3\pi/4$.



5. (a) $\mathcal{Z}$ is a linear function of jointly Gaussian random variables and hence is itself a Gaussian random variable with mean $E[\mathcal{Z}] = E[5\mathcal{X} + \mathcal{Y}] = 14$ and variance $\sigma_{\mathcal{Z}}^2 = 5^2 \cdot \sigma_{\mathcal{X}}^2 + 1^2 \cdot \sigma_{\mathcal{Y}}^2 + 2 \cdot 5 \cdot 1 \cdot \rho\sigma_{\mathcal{X}}\sigma_{\mathcal{Y}} = 100 + 16 + 5 = 121$.
 Hence, $f_{\mathcal{Z}}(\alpha) = (11\sqrt{2\pi})^{-1/2} \exp(-(\alpha - 14)^2/242)$, $-\infty < \alpha < \infty$.
- (b) $P\{\mathcal{Y} > 3\mathcal{X}\} = P\{3\mathcal{X} - \mathcal{Y} < 0\}$ where $3\mathcal{X} - \mathcal{Y}$ is a Gaussian random variable with mean $3 \cdot 0 - 14 = -14$ and variance $3^2 \cdot \sigma_{\mathcal{X}}^2 + 1^2 \cdot \sigma_{\mathcal{Y}}^2 - 2 \cdot 3 \cdot 1 \cdot \rho\sigma_{\mathcal{X}}\sigma_{\mathcal{Y}} = 36 + 16 - 3 = 49$.
 Hence, $P\{\mathcal{Y} > 3\mathcal{X}\} = \Phi(0 - (-14))/7 = \Phi(2) = 0.9772$.