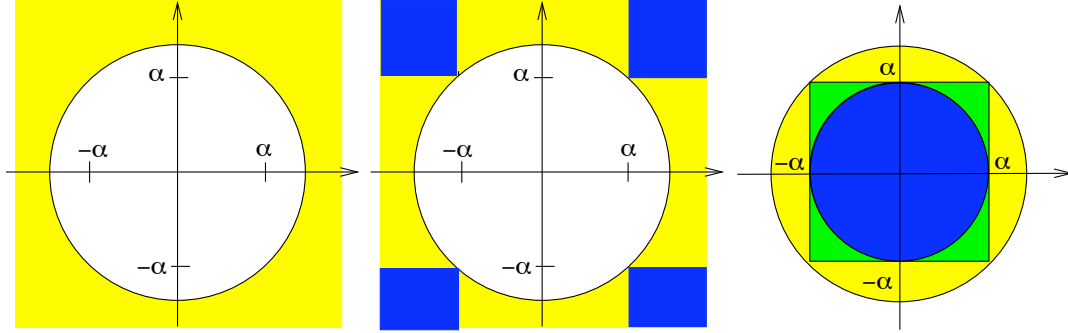


ECE 313: Solutions to Problem Set 12

1. (a) $f_{\mathcal{X},\mathcal{Y}}(u, v) = f_{\mathcal{X}}(u)f_{\mathcal{Y}}(v) = \frac{1}{2\pi} \exp\left[-\frac{u^2 + v^2}{2}\right]$.
- (b) The region over which the joint pdf must be integrated in order to find $P\{\mathcal{X}^2 + \mathcal{Y}^2 > 2\alpha^2\}$ is shown in the left-hand figure below.



The volume outside the circle of radius $\sqrt{2}\alpha$ can be found by changing to polar coordinates. We have $P\{\mathcal{X}^2 + \mathcal{Y}^2 > 2\alpha^2\} = \int_{r=\sqrt{2}\alpha}^{\infty} \int_{\theta=0}^{2\pi} \frac{1}{2\pi} \exp\left(-\frac{r^2}{2}\right) r d\theta dr = \int_{r=\sqrt{2}\alpha}^{\infty} r \exp\left(-\frac{r^2}{2}\right) dr = -\exp\left(-\frac{r^2}{2}\right) \Big|_{\sqrt{2}\alpha}^{\infty} = \exp(-\alpha^2)$.

- (c) $\mathcal{Z} = \mathcal{X}^2 + \mathcal{Y}^2$. From part (b), we get that $P\{\mathcal{Z} > \beta\} = 1 - F_{\mathcal{Z}}(\beta) = \exp(-\beta/2)$ for $\beta \geq 0$. Hence, $f_{\mathcal{Z}}(\beta) = \frac{1}{2} \exp(-\beta/2)$ for $\beta \geq 0$. This is an exponential density with parameter $\frac{1}{2}$.
- (d) $P\{|\mathcal{X}| > \alpha\} = 2Q(\alpha)$. Hence, $P\{|\mathcal{X}| > \alpha, |\mathcal{Y}| > \alpha\} = P\{|\mathcal{X}| > \alpha\}P\{|\mathcal{Y}| > \alpha\} = 4Q^2(\alpha)$.
- (e) From the middle figure above, it is obvious that

$$P\{|\mathcal{X}| > \alpha, |\mathcal{Y}| > \alpha\} = 4Q^2(\alpha) < P\{\mathcal{X}^2 + \mathcal{Y}^2 > 2\alpha^2\} = \exp(-\alpha^2) \text{ for } \alpha > 0.$$

- (f) Taking square roots on both sides, we get $Q(x) < \frac{1}{2} \exp(-x^2/2)$ for $x > 0$.
- (g) $P\{|\mathcal{X}| < \alpha, |\mathcal{Y}| < \alpha\} = P\{|\mathcal{X}| < \alpha\}P\{|\mathcal{Y}| < \alpha\} = [1 - 2Q(\alpha)]^2$ is the probability that the random point lies inside the square of side 2α . As shown in the right-hand figure, this square is inscribed by the circle of radius α and circumscribed by the circle of radius $\sqrt{2}\alpha$. Hence,

$$P\{\mathcal{X}^2 + \mathcal{Y}^2 \leq \alpha^2/2\} < P\{|\mathcal{X}| < \alpha, |\mathcal{Y}| < \alpha\} < P\{\mathcal{X}^2 + \mathcal{Y}^2 < \alpha^2\}.$$

These inequalities are equivalent to the result: $\exp(-\alpha^2) < 4Q(\alpha) - 4Q^2(\alpha) < \exp(-\alpha^2/2)$. Now, $4Q(\alpha) - 4Q^2(\alpha) < 4Q(\alpha)$, and therefore we get $Q(x) > \frac{1}{4} \exp(-x^2)$ for $x > 0$. Note that $Q(0) = \frac{1}{2}$ while the bound equals $\frac{1}{4}$ at $x = 0$. The upper bound can also be obtained from the above result. Note that since $Q(\alpha) \leq \frac{1}{2}$ for $\alpha > 0$, it follows that $4Q(\alpha) - 4Q^2(\alpha) = 4Q(\alpha)[1 - Q(\alpha)] > 2Q(\alpha)$ for $\alpha > 0$. Hence, $\exp(-\alpha^2/2) > 4Q(\alpha) - 4Q^2(\alpha) > 2Q(\alpha)$ etc.

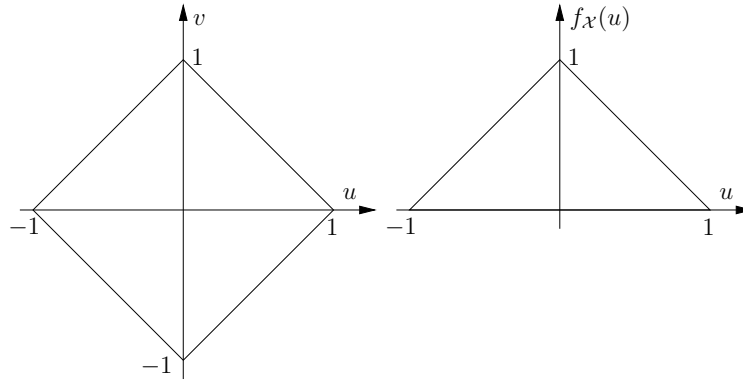
2. (a) The joint pdf solid is a unit-radius cylinder of height c with a unit-radius cone of height c scooped out of it, leaving a volume of $\pi c - \frac{1}{3}\pi c = \frac{2\pi}{3}c$. Hence, $c = \frac{3}{2\pi}$. Alternatively, converting to polar coordinates:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{\mathcal{X},\mathcal{Y}}(u, v) du dv = \int_{\theta=0}^{2\pi} \int_{r=0}^1 cr \cdot r dr d\theta = \frac{2\pi c}{3} = 1 \Rightarrow c = \frac{3}{2\pi}.$$

- (b) Using LOTUS and converting to polar coordinates again,

$$\mathbb{E}[\sqrt{\mathcal{X}^2 + \mathcal{Y}^2}] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sqrt{u^2 + v^2} \cdot f_{\mathcal{X},\mathcal{Y}}(u, v) du dv = \frac{3}{2\pi} \int_{\theta=0}^{2\pi} \int_{r=0}^1 r^3 dr d\theta = \frac{3}{4}.$$

3. Since the joint pdf is uniform over the square region of area 2 shown in the left-hand figure below, we have that $f_{\mathcal{X},\mathcal{Y}}(u, v) = \frac{1}{2}$ over the region.



- (a) The value of the marginal pdf of $\mathcal{X}$ at u is the *area of the cross-section* of the pdf solid at u . Since the cross-section is a rectangle of height $\frac{1}{2}$ and base $2(1 - |u|)$ for $|u| \leq 1$, we get $f_{\mathcal{X}}(u) = \begin{cases} 1 - |u|, & -1 \leq u \leq 1, \\ 0, & \text{otherwise.} \end{cases}$. The pdf is sketched in the right-hand figure above. It should be obvious by inspection that we have obtained a valid pdf.

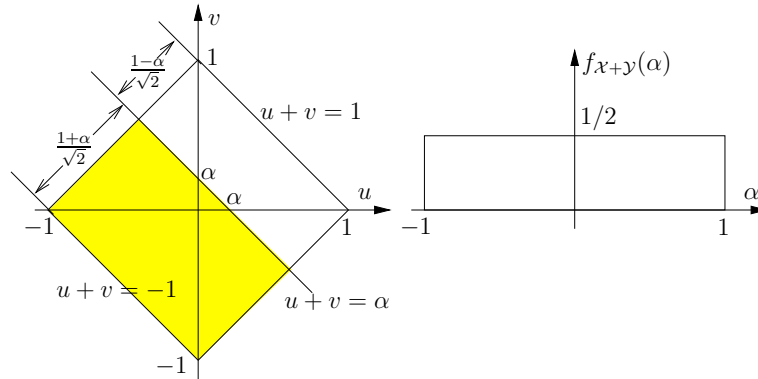
By symmetry, $\mathcal{Y}$ has the same pdf, that is, $f_{\mathcal{Y}}(v) = \begin{cases} 1 - |v|, & -1 \leq v \leq 1, \\ 0, & \text{otherwise.} \end{cases}$

Since the support of $f_{\mathcal{X},\mathcal{Y}}(u, v)$ is not a rectangle with sides parallel to the axes, the *eyeball* test tells us that $\mathcal{X}$ and $\mathcal{Y}$ are not independent random variables. Alternatively, $f_{\mathcal{X},\mathcal{Y}}(u, v) \neq f_{\mathcal{X}}(u)f_{\mathcal{Y}}(v)$ for all u and v , and so $\mathcal{X}$ and $\mathcal{Y}$ are not independent random variables.

- (b) By symmetry of the pdf of $\mathcal{X}$, $E[\mathcal{X}] = 0$. It follows that $\text{var}(\mathcal{X}) = E[\mathcal{X}^2] - (E[\mathcal{X}])^2 = E[\mathcal{X}^2]$, and we get that

$$\text{var}(\mathcal{X}) = \int_{-\infty}^{\infty} u^2 f_{\mathcal{X}}(u) du = \int_{-1}^1 u^2 (1 - |u|) du = 2 \int_0^1 (1 - u) u^2 du = 2 \left[\frac{u^3}{3} \Big|_0^1 - \frac{u^4}{4} \Big|_0^1 \right] = \frac{1}{6}.$$

- (c) The random variable $\mathcal{Z} = \mathcal{X} + \mathcal{Y}$ can take on values in the interval $[-1, +1]$ and hence $F_{\mathcal{Z}}(\alpha) = 0$ for $\alpha < -1$ and $F_{\mathcal{Z}}(\alpha) = 1$ for $\alpha > 1$. For $-1 \leq \alpha \leq +1$, $F_{\mathcal{Z}}(\alpha) = P\{\mathcal{Z} \leq \alpha\} = P\{\mathcal{X} + \mathcal{Y} \leq \alpha\}$ is

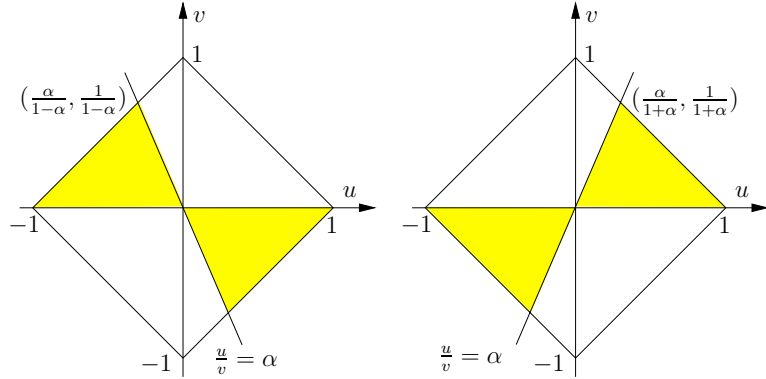


the total probability mass (that is, volume under pdf surface) in the shaded region in the left-hand figure above. Since the region has area $\sqrt{2} \times (1 + \alpha)/\sqrt{2}$ and the pdf has value $\frac{1}{2}$, we get that

$$F_{\mathcal{Z}}(\alpha) = P\{\mathcal{Z} \leq \alpha\} = \begin{cases} 0, & \alpha < -1, \\ \frac{1+\alpha}{2}, & -1 \leq \alpha \leq 1, \\ 1, & \alpha > 1, \end{cases} \text{ and } f_{\mathcal{Z}}(\alpha) = \frac{d}{d\alpha} F_{\mathcal{Z}}(\alpha) = \begin{cases} \frac{1}{2}, & -1 \leq \alpha \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

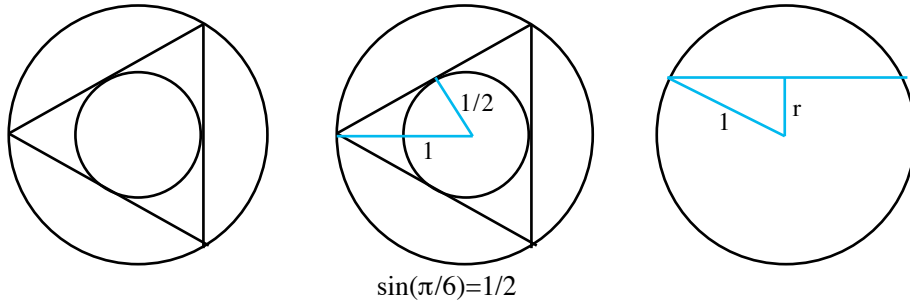
The pdf $f_{\mathcal{Z}}(\alpha) = f_{\mathcal{X}+\mathcal{Y}}(\alpha)$ is sketched in the right-hand figure above. Note that $\mathcal{Z} \sim \text{Uniform}[-1, 1]$.

- (d) Since $\mathcal{Y}$ can take on value 0, the ratio $\mathcal{X}/\mathcal{Y} = \mathcal{W}$ can take on all values in $(-\infty, \infty)$. Suppose that $\alpha < 0$. Then, $F_{\mathcal{W}}(\alpha) = P\{\mathcal{W} \leq \alpha\} = P\{(\mathcal{X}, \mathcal{Y}) \in A\}$ where A is the *shaded* region in the left-hand figure below. Since the two triangles together have total area $1/(1-\alpha)$ and the joint pdf has value $\frac{1}{2}$, we see that $F_{\mathcal{W}}(\alpha) = \frac{1}{2(1-\alpha)}$ which increases from 0 at $\alpha = -\infty$ to $\frac{1}{2}$ at $\alpha = 0$.



On the other hand, if $\alpha > 0$, then $P\{\mathcal{W} > \alpha\} = 1 - F_{\mathcal{W}}(\alpha) = P\{(\mathcal{X}, \mathcal{Y}) \in B\}$ where B is the shaded region in the right-hand figure above. Since the two triangles together have total area $1/(1+\alpha)$ and the joint pdf has value $\frac{1}{2}$, we see that $P\{\mathcal{W} > \alpha\} = \frac{1}{2(1+\alpha)}$ which decreases from $\frac{1}{2}$ at $\alpha = 0$ to 0 at $\alpha = \infty$. Differentiating, we get that $f_{\mathcal{W}}(\alpha) = \frac{1}{2(1+|\alpha|)^2}$, $-\infty < \alpha < \infty$.

4. The pdf of $(\mathcal{X}, \mathcal{Y})$ is uniformly distributed on the unit disc, and thus has value π^{-1} .



- (a) The circle inscribed in the triangle has radius $1/2$ (cf. Ross, p. 217). If the midpoint of the chord is outside the circle, then the length $\mathcal{L}$ of the random chord is smaller than the side of the equilateral triangle. Hence, $P\{\mathcal{L} > \frac{1}{2}\} = \left[\pi \left(\frac{1}{2}\right)^2\right] \pi^{-1} = \frac{1}{4}$
- (b) The length of a chord at distance r from the center is $2\sqrt{1-r^2}$. Hence, $\mathcal{L} = 2\sqrt{1-\mathcal{R}^2} = 2\sqrt{1-\mathcal{X}^2-\mathcal{Y}^2}$ and $F_{\mathcal{L}}(\alpha) = P\{\mathcal{L} \leq \alpha\} = P\{\mathcal{X}^2 + \mathcal{Y}^2 \geq 1 - \alpha^2/4\} = 1 - \pi(1 - \alpha^2/4)\pi^{-1} = \alpha^2/4$. From this we get that $f_{\mathcal{L}}(\alpha) = \alpha/2$ for $0 \leq \alpha \leq 2$, and $f_{\mathcal{L}}(\alpha) = 0$ otherwise.

(c) $E[\mathcal{L}] = \int_0^2 \alpha \cdot \frac{\alpha}{2} d\alpha = \frac{4}{3}$

5. The equation has real roots if $\mathcal{X}^2 \geq 4\mathcal{Y}$. We have that

$$P\{\mathcal{X}^2 \geq 4\mathcal{Y}\} = P\{\mathcal{Y} \leq \mathcal{X}^2/4\} = \int_{u=0}^{u=1} \int_{v=0}^{v=u^2/4} dv du = \int_0^1 \frac{u^2}{4} du = \frac{1}{12}.$$