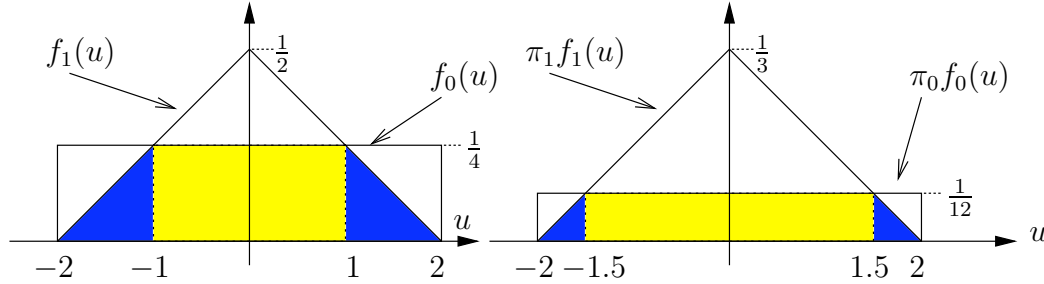


ECE 313: Solutions to Problem Set 11

1. (a) The easiest way to solve this problem is to sketch the two pdfs as shown in the left-hand figure below.



It is obvious that the maximum-likelihood decision is in favor of H_1 if $|\mathcal{X}| < 1$, and hence $x = 0$, $\eta = 1$. By inspection, we get that $P_{FA} = 2 \times \frac{1}{4} = \frac{1}{2}$ while $P_{MD} = 2 \times \left(\frac{1}{2} \times 1 \times \frac{1}{4}\right) = \frac{1}{4}$.

The graphically-challenged can proceed as follows.

For $-2 < u < 2$, the likelihood ratio is $\Lambda(u) = \frac{f_1(u)}{f_0(u)} = \frac{0.25(2 - |u|)}{0.25} = 2 - |u|$.

When $\mathcal{X} = u$ is the observation, the *maximum-likelihood* decision rule decides in favor of H_1 if $\Lambda(u) > 1$. Hence $\Gamma_1 = \{u : |u| < 1\}$ and $\Gamma_0 = \{u : 1 < |u| < 2\}$, that is, the ML decision rule is that if $|\mathcal{X}| > 1$, the decision is that H_0 is the true hypothesis. Thus, we have $x = 0$, and $\eta = 1$.

$$P_{FA} = \int_{\Gamma_1} f_0(u) du = \int_{-1}^1 \frac{1}{4} du = \frac{1}{2}.$$

$$P_{MD} = \int_{\Gamma_0} f_1(u) du = 2 \int_1^2 \frac{1}{4}(2 - u) du = \frac{1}{2} \left(2u - \frac{u^2}{2}\right) \Big|_1^2 = \frac{1}{4}.$$

- (b) The probability of error of the ML decision rule is

$$P(E) = \pi_0 P_{FA} + \pi_1 P_{MD} = \frac{1}{3} \times \frac{1}{2} + \frac{2}{3} \times \frac{1}{4} = \frac{1}{3}.$$

- (c) Sketching $\pi_0 f_0(u)$ and $\pi_1 f_1(u)$ as in the right-hand figure above, we easily see that the MAP decision is in favor of H_1 if $|\mathcal{X}| < 1.5$, and hence $x = 0$, $\xi = 1.5$. By inspection, we get that $\pi_0 P_{FA} = 3 \times \frac{1}{12} = \frac{1}{4} = \frac{1}{3} \times \frac{3}{4}$ while $\pi_1 P_{MD} = 2 \times \left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{12}\right) = \frac{1}{24} = \frac{2}{3} \times \frac{1}{16}$, that is, $P_{FA} = \frac{3}{4}$, and $P_{MD} = \frac{1}{16}$. $P(E) = \pi_0 P_{FA} + \pi_1 P_{MD} = \frac{1}{4} + \frac{1}{24} = \frac{7}{24} < \frac{1}{3}$, where $\frac{1}{3}$ is the error probability of the ML rule (with the same *a priori* probabilities) that we found in part (c).

Without using any graphical aids, we have that when $\mathcal{X} = u$ is the observation, the MAP decision rule decides in favor of H_1 if $\Lambda(u) = 2 - |u| > \pi_0/\pi_1 = 1/2$. Hence, $\Gamma_1 = \{u : |u| < \frac{3}{2}\}$ and $\Gamma_0 = \{u : \frac{3}{2} < |u| < 2\}$ for the MAP decision rule. Once

again, $x = 0$ while $\xi = \frac{3}{2}$. We get $P_{FA} = \int_{\Gamma_1} f_0(u) du = \int_{-\frac{3}{2}}^{\frac{3}{2}} \frac{1}{4} du = \frac{3}{4}$.

$$P_{MD} = \int_{\Gamma_0} f_1(u) du = 2 \int_{\frac{3}{2}}^2 \frac{1}{4}(2 - u) du = \frac{1}{2} \left(2u - \frac{u^2}{2}\right) \Big|_{\frac{3}{2}}^2 = \frac{1}{16}.$$

$$\text{Hence, } P(E) = \pi_0 \cdot P_{FA} + \pi_1 \cdot P_{MD} = \frac{1}{3} \times \frac{3}{4} + \frac{2}{3} \times \frac{1}{16} = \frac{1}{4} + \frac{1}{24} = \frac{7}{24} < \frac{1}{3}.$$

- (d) The MAP decision rule compares the likelihood ratio $\Lambda(u) = 2 - |u|$ and decides in favor of H_0 if $\Lambda(u) < \pi_0/\pi_1$. Since $\Lambda(u)$ takes on values in $(0, 2]$, it cannot exceed π_0/π_1 if $\pi_0 > \frac{2}{3}$. Thus, the MAP decision rule always decides in favor of H_0 if $\pi_0 > 2/3$, and achieves average error probability $P(E) = \pi_1 < 1/3$.
- (e) On the other hand, there is *no* value of $\pi_0, 0 < \pi_0 < 1$ for which the MAP rule *always* decide in favor of H_1 , because no matter how large π_1 is, the ratio $\pi_0/\pi_1 > 0$. Thus, for *some* values of $\mathcal{X}$, the likelihood ratio will be smaller than the threshold, and the decision will be in favor of H_0 .

2. (a) It is easy to see that

$$\begin{aligned}\{Y > T\} = & (\{\mathcal{X}_1 > T\} \cap \{\mathcal{X}_2 > T\} \cap \{\mathcal{X}_3 > T\}) \\ & \cup (\{\mathcal{X}_1 > T\} \cap \{\mathcal{X}_2 > T\} \cap \{\mathcal{X}_3 \leq T\}) \\ & \cup (\{\mathcal{X}_1 > T\} \cap \{\mathcal{X}_2 \leq T\} \cap \{\mathcal{X}_3 > T\}) \\ & \cup (\{\mathcal{X}_1 \leq T\} \cap \{\mathcal{X}_2 > T\} \cap \{\mathcal{X}_3 > T\})\end{aligned}$$

- (b) Since $P\{\mathcal{X}_i > T\} = \exp(-\lambda T)$, we get that

$$P\{\mathcal{Y} > T\} = (\exp(-\lambda T))^3 + 3(\exp(-\lambda T))^2(1 - \exp(-\lambda T)) = 3\exp(-2\lambda T) - 2\exp(-3\lambda T).$$

(c) $E[\mathcal{Y}] = \int_0^\infty P\{\mathcal{Y} > T\} dT = \int_0^\infty 3\exp(-2\lambda T) - 2\exp(-3\lambda T) dT = \frac{3}{2\lambda} - \frac{2}{3\lambda} = \frac{5}{6}\lambda^{-1}.$

(d) $p_{\mathcal{Y}}(T) = \frac{d}{dT}P\{\mathcal{Y} \leq T\} = \frac{d}{dT}(1 - P\{\mathcal{Y} > T\}) \Rightarrow p_{\mathcal{Y}}(T) = 6\lambda\exp(-2\lambda T) - 6\lambda\exp(-3\lambda T),$

Hence, the hazard rate is $h_{\mathcal{Y}}(t) = \frac{p_{\mathcal{Y}}(t)}{P\{\mathcal{Y} > t\}} = \frac{6\lambda[1 - \exp(-\lambda t)]}{3 - 2\exp(-\lambda t)}$. Note that the hazard rate is 0 at $t = 0$ but increases with t , having value λ at $t = \lambda^{-1} \ln(4/3) = T_0$, and then approaching 2λ asymptotically as $t \rightarrow \infty$.

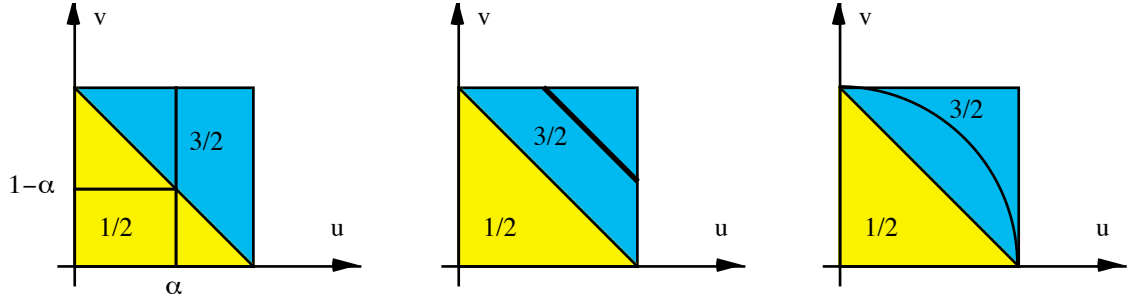
- (e) Let $p = \exp(-\lambda T)$. Then, $P\{\mathcal{Y} > T\} = 3p^2 - 2p^3 = \frac{1}{2}$, which has only one solution $p = 1/2$ in the range $0 \leq p \leq 1$. Thus, the median value of $\mathcal{Y}$ is the solution T_{median} to $\exp(-\lambda T_{\text{median}}) = 1/2$, giving $T_{\text{median}} = \lambda^{-1} \ln 2$ which is the same as the median lifetime of a single module. Note that $T_0 < T_{\text{median}}$.
- (f) The MTBF λ^{-1} of the single module is larger (!) than the MTBF of the TMR system. The median lifetimes are the same. The TMR system *does not* result in a more reliable system judging by MTBF and median lifetimes as the sole criteria.
- (g) If $\lambda = -\ln 0.999 \approx 0.0010$, then a module has an average lifetime of ≈ 999.5 time units. The TMR system and a single module have median lifetimes of ≈ 692.8 time units. The parameter T_0 is approximately 287.5. of a module and the $T \exp(-\lambda) = 0.999$. We have that $P\{\mathcal{X}_1 > 1\} = \exp(-\lambda) = 0.999$ while $P\{\mathcal{Y} > 1\} = 3(0.999)^2 - 2(0.999)^3 = 0.999997002$ so that the TMR system is likely to provide much more reliability for one unit of time. More generally,

$$P\{\mathcal{Y} > T\} = 3p^2 - 2p^3 > P\{\mathcal{X} > T\} = p \text{ for } \frac{1}{2} < p < 1, \text{ that is, for } T < T_{\text{median}} \text{ while}$$

$P\{\mathcal{Y} > T\} < P\{\mathcal{X} > T\} = p$ for $0 < p < \frac{1}{2}$, that is, for $T > T_{\text{median}}$. So, the probability of a TMR system working for T time units is larger than the probability of a single module working for T time units, *provided* $T < T_{\text{median}}$. However, for $T_0 < T < T_{\text{median}}$, the TMR system has a larger hazard rate than a single module.

- (h) The largest value of T for which $P\{\mathcal{Y} > T\} \geq 0.999$ is $-\lambda^{-1} \ln 0.98163 \approx 18.53$. Thus, the TMR system can be expected to work (with 99.9% reliability) for more than 18 units of time whereas the single module provides this level of reliability for just one unit of time. It is in matters such as these that the TMR systems shines ...not in terms of comparisons of MTBF or median lifetimes.

3. The joint pdf is as shown in the figure below.



- (a) From the left-hand figure above, $f_{\mathcal{X}}(\alpha) = 0$ for $\alpha < 0$ or $\alpha > 1$, while for any $\alpha, 0 \leq \alpha \leq 1$,

$$f_{\mathcal{X}}(\alpha) = \int_{-\infty}^{\infty} f_{\mathcal{X},\mathcal{Y}}(\alpha, v) dv = \int_0^{1-\alpha} \frac{1}{2} dv + \int_{1-\alpha}^1 \frac{3}{2} dv = \frac{1}{2}(1 - \alpha) + \frac{3}{2}\alpha = \frac{1}{2} + \alpha.$$

- (b) When the pdf has constant value over a region, we can find the probability that the random point lies in that region by finding the area of the region and multiplying by the pdf value.

$$\text{Thus, } P\{\mathcal{X} + \mathcal{Y} \leq 3/2\} = 1 - P\{\mathcal{X} + \mathcal{Y} \geq 3/2\} = 1 - \frac{3}{2} \times \left[\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \right] = \frac{13}{16}$$

$$\text{and } P\{\mathcal{X}^2 + \mathcal{Y}^2 \geq 1\} = \frac{3}{2} \times \left[1 - \frac{\pi}{4} \right] = \frac{3}{2} - \frac{3\pi}{8}.$$

4. The pdf is nonzero over the shaded region in the left-hand figure shown at the bottom of the page.

- (a) , (b) From the figure, we get that for $v > 0$,

$$f_{\mathcal{Y}}(v) = \int_{u=-v}^{u=+v} c(v^2 - u^2) \exp(-v) du = c \left[v^2 u - \frac{u^3}{3} \right] \exp(-v) \Big|_{u=-v}^{u=+v} = c \left(\frac{4}{3} \right) v^3 \exp(-v)$$

which is of the form of a *gamma* pdf with parameters $(4, 1)$.

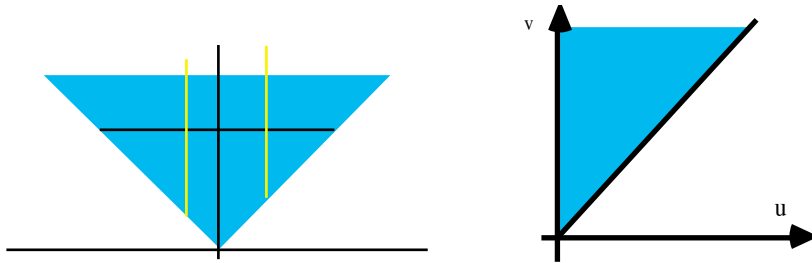
Thus, $4c/3 = 1/\Gamma(4) = 1/3 \Rightarrow c = 1/8$.

On the other hand, for $u > 0$, $f_{\mathcal{X}}(u) = \int_{v=u}^{\infty} \frac{1}{8}(v^2 - u^2) \exp(-v) dv = \frac{1}{4}(1 + u) \exp(-u)$,

while if $u < 0$, the limits are $v = -u$ and ∞ .

Consequently, $f_{\mathcal{X}}(u) = \frac{1}{4}(1 + |u|) \exp(-u)$, $-\infty < u < \infty$.

- (c) The pdf of $\mathcal{X}$ is an even function of u and $\int_0^{\infty} f_{\mathcal{X}}(u) du$ is finite. Hence, $E[\mathcal{X}] = 0$.



5. (a) The joint pdf is nonzero on the shaded region shown in the right-hand figure above.

- (b) For $u > 0$, $f_{\mathcal{X}}(u) = \int_v^\infty 2 \exp(-u - v) dv = 2 \exp(-2u)$ and $f_{\mathcal{X}}(u) = 0$ for $u < 0$.
 For $v > 0$, $f_{\mathcal{Y}}(v) = \int_0^v 2 \exp(-u - v) du = 2 \exp(-v) - 2 \exp(-2v)$ and $f_{\mathcal{Y}}(v) = 0$ for $v < 0$.
- (c) $P\{\mathcal{Y} > 3\mathcal{X}\} = \int_{u=0}^\infty \int_{v=3u}^\infty 2(\exp(-u - v)) dv du = \int_{u=0}^\infty 2 \exp(-4u) du = \frac{1}{2}$.
- (d) For $\alpha > 0$,

$$P\{\mathcal{X} + \mathcal{Y} \leq \alpha\} = \int_{u=0}^{\alpha/2} \int_{v=u}^{\alpha-u} 2e^{-u-v} dv du = \int_{u=0}^{\alpha/2} 2e^{-u} [e^{(-u)} - e^{(-\alpha+u)}] du$$

$$= 1 - \alpha \exp(-\alpha) - \exp(-\alpha).$$