

ECE 313: Solutions to Problem Set 10

In Problems 1 and 2, $\phi(u)$ and $\Phi(u)$ denote the pdf and CDF of the standard Gaussian random variable.

1. (a) $\mathbb{E}[\mathcal{X}^{2k+1}] = \int_{-\infty}^{\infty} u^{2k+1} \phi(u) du = 0$ because the integrand is an odd function.

(b) Since $d\phi(u) = -u\phi(u)du$, we can integrate by parts to get

$$\begin{aligned} \mathbb{E}[\mathcal{X}^{2k}] &= \int_{-\infty}^{\infty} u^{2k} \phi(u) du = \int_{-\infty}^{\infty} u^{2k-1} \cdot u\phi(u) du = - \int_{-\infty}^{\infty} (u^{2k-1}) \cdot d\phi(u) \\ &= -u^{2k-1} \phi(u) \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} (2k-1) u^{2k-2} \phi(u) du = 0 + (2k-1) \int_{-\infty}^{\infty} u^{2k-2} \phi(u) du \\ &= (2k-1) \mathbb{E}[\mathcal{X}^{2k-2}] = (2k-1)(2k-3) \mathbb{E}[\mathcal{X}^{2k-4}] = \dots = (2k-1)(2k-3) \dots 3 \cdot 1 \cdot \mathbb{E}[\mathcal{X}^0] \\ &= (2k-1)(2k-3) \dots 3 \cdot 1 = \frac{(2k)!}{2^k k!}. \end{aligned}$$

(c) Let $\mathcal{Y} = \frac{\mathcal{X}^2}{R}$. Then, $\mathcal{Y}$ takes on values in $[0, \infty)$ and thus $F_{\mathcal{Y}}(v) = 0$ for $v < 0$. For $v \geq 0$, $F_{\mathcal{Y}}(v) = P\{\mathcal{Y} \leq v\} = P\{\frac{\mathcal{X}^2}{R} \leq v\} = P\{\mathcal{X}^2 \leq vR\} = P\{-\sqrt{vR} \leq \mathcal{X} \leq \sqrt{vR}\} = \Phi(\sqrt{vR}) - \Phi(-\sqrt{vR})$ since $\mathcal{X}$ is a standard Gaussian random variable. Thus, $f_{\mathcal{Y}}(v) = 0$ for $v < 0$ while for $v \geq 0$, $f_{\mathcal{Y}}(v) = \frac{d}{dv} F_{\mathcal{Y}}(v) = \frac{1}{2} \sqrt{\frac{R}{v}} \left[\phi(\sqrt{vR}) + \phi(-\sqrt{vR}) \right] = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{R}{v}} \exp(-\frac{1}{2}vR) \sim \text{Gamma}(\frac{1}{2}, \frac{R}{2})$ pdf.

2. (a) Since $\phi(\cdot)$ is an even function, $1 - \Phi(x) = \int_{-\infty}^{-x} \frac{1}{t} t\phi(t) dt = \int_{\infty}^x \frac{-1}{u} u\phi(-u) du = \int_x^{\infty} \frac{1}{u} u\phi(u) du$. Integrating by parts as in Problem 1(b) above, we have that for $x > 0$,

$$1 - \Phi(x) = - \int_x^{\infty} \frac{1}{u} d\phi(u) = \frac{-1}{u} \phi(u) \Big|_x^{\infty} - \int_x^{\infty} \frac{1}{u^2} \cdot \phi(u) du = \frac{1}{x} \phi(x) - \int_x^{\infty} \frac{1}{u^2} \cdot \phi(u) du.$$

Since the integrand of the rightmost integral in the line above is positive, so is the integral. Even though we do not know the exact value of the integral, we can nonetheless deduce that $1 - \Phi(x) < x^{-1} \phi(x)$ for $x > 0$.

(b) Next, writing the integrand above as $\frac{-1}{u^3} \cdot (-u\phi(u))$ and repeating the integration by parts and the argument about the value of an integral being positive, we get

$$1 - \Phi(x) = \frac{1}{x} \phi(x) - \int_x^{\infty} \frac{-1}{u^3} \cdot (-u\phi(u)) du = \frac{1}{x} \phi(x) - \frac{-1}{u^3} \phi(u) \Big|_x^{\infty} + \int_x^{\infty} \frac{3}{u^4} \cdot \phi(u) du$$

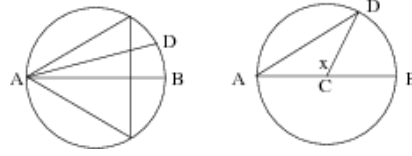
showing that $1 - \Phi(x) > (x^{-1} - x^{-3}) \phi(x)$ for $x > 0$.

These results were proved in the note *The Complementary Unit Gaussian Distribution Function $Q(x)$* available on the COMPASS web page for ECE 313.

3. The probability that a device operates for at least 15 hours is $P\{\mathcal{X} > 15\} = \exp(-15)$. In general, it is not possible to compute the probability that at least three of the six devices will operate for at least 15 hours from this information. However, if we *assume* that all six devices operate independently and so their failures are independent of each other, then $\mathcal{Y}$, the number of devices that are operational for at least 15 hours is a binomial random variable with parameters $(6, \exp(-15))$, and

$$P\{\mathcal{Y} \geq 3\} = \sum_{i=3}^6 \binom{6}{i} [\exp(-15)]^i (1 - \exp(-15))^{6-i} \approx 5.7250 \times 10^{-19}.$$

4. (a) $\mathcal{X}$ is uniformly distributed on $[0, 2\pi)$. From the diagram below, it should be obvious that the probability that the random chord is longer than the side of the inscribed equilateral triangle is $P\{2\pi/3 < \mathcal{X} < 4\pi/3\} = \frac{1}{3}$.



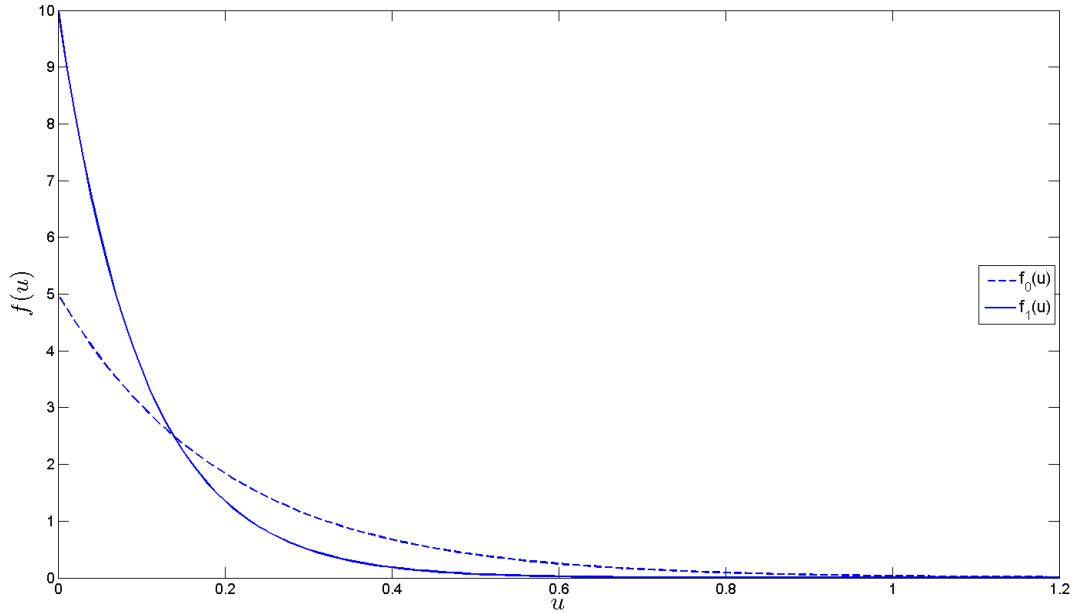
- (b) Since the circle has radius 1, an arc of length $\mathcal{X}$ subtends angle $\mathcal{X}$ at the center of the circle. Furthermore, the length $\mathcal{L}$ of the chord is $2 \sin(\mathcal{X}/2)$, increasing from 0 when $\mathcal{X} = 0$ to 2 when $\mathcal{X} = \pi$ and decreasing back to 0 at $\mathcal{X} = 2\pi$. For any $x, 0 < x < 2$,

$$F_{\mathcal{L}}(x) = P\{\mathcal{L} \leq x\} = P\{2 \sin(\mathcal{X}/2) \leq x\} = 2 \cdot P\{0 \leq \mathcal{X} \leq 2 \arcsin(x/2)\} = \left(\frac{2}{\pi}\right) \arcsin\left(\frac{x}{2}\right)$$

Hence,

$$f_{\mathcal{L}}(x) = \frac{d}{dx} F_{\mathcal{L}}(x) = \begin{cases} \frac{1}{\pi \sqrt{1 - (x/2)^2}}, & 0 \leq x \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

5. (a) The pdfs are as shown below. Note that it is immediately obvious that small values of $\mathcal{X}$ result in decisions in favor of H_1 and large values of $\mathcal{X}$ result in decisions in favor of H_0 .



- (b) $\Lambda(u) = \frac{f_1(u)}{f_0(u)} = \frac{10 \cdot \exp(-10u)}{5 \cdot \exp(-5u)} = 2 \cdot \exp(-5u)$ which has value 2 at $u = 0$ and decays away to 0 as $u \rightarrow \infty$. Note that $\Lambda(u) > 1$ for $u < 0.2 \ln 2$. Thus, the likelihood ratio test is equivalent to deciding in favor of H_1 if the observed value of $\mathcal{X}$ is *smaller* than the threshold $0.2 \ln 2$. Equivalently, $\Gamma_1 = (0, 0.2 \ln 2)$, $\Gamma_0 = (0.2 \ln 2, \infty)$.

$$(c) P_{\text{FA}} = \int_{\Gamma_1} f_0(u) du = \int_0^{0.2 \ln 2} 5 \cdot \exp(-5u) du = -\exp(-5u) \Big|_0^{0.2 \ln 2} = -\frac{1}{2} - (-1) = \frac{1}{2}.$$

$$P_{\text{MD}} = \int_{\Gamma_0} f_1(u) du = \int_{0.2 \ln 2}^{\infty} 10 \cdot \exp(-10u) du = -\exp(-10u) \Big|_{0.2 \ln 2}^{\infty} = 0 - (-\exp(-2 \ln 2)) = \frac{1}{4}.$$

(d) $\Lambda(u) = 2 \cdot \exp(-5u) > \frac{\pi_0}{\pi_1}$ for $u < 0.2 \ln \left(\frac{2\pi_1}{\pi_0} \right) = 0.2 \ln 2 + 0.2 \ln \left(\frac{\pi_1}{\pi_0} \right) = \xi$. Thus, the minimum-error-probability decision rule is equivalent to deciding in favor of H_1 if the observed value of $\mathcal{X}$ is smaller than ξ . Note that $\xi < 0$ if $\pi_0 > 2\pi_1$, that is, if $\pi_0 > 2/3$.

(e) If $\pi_0 = 1/3$, then $\xi = 0.2 \ln 4$. Hence,

$$P_{\text{FA}} = \int_0^{0.2 \ln 4} 5 \cdot \exp(-5u) du = -\exp(-5u) \Big|_0^{0.2 \ln 4} = -\frac{1}{4} - (-1) = \frac{3}{4}.$$

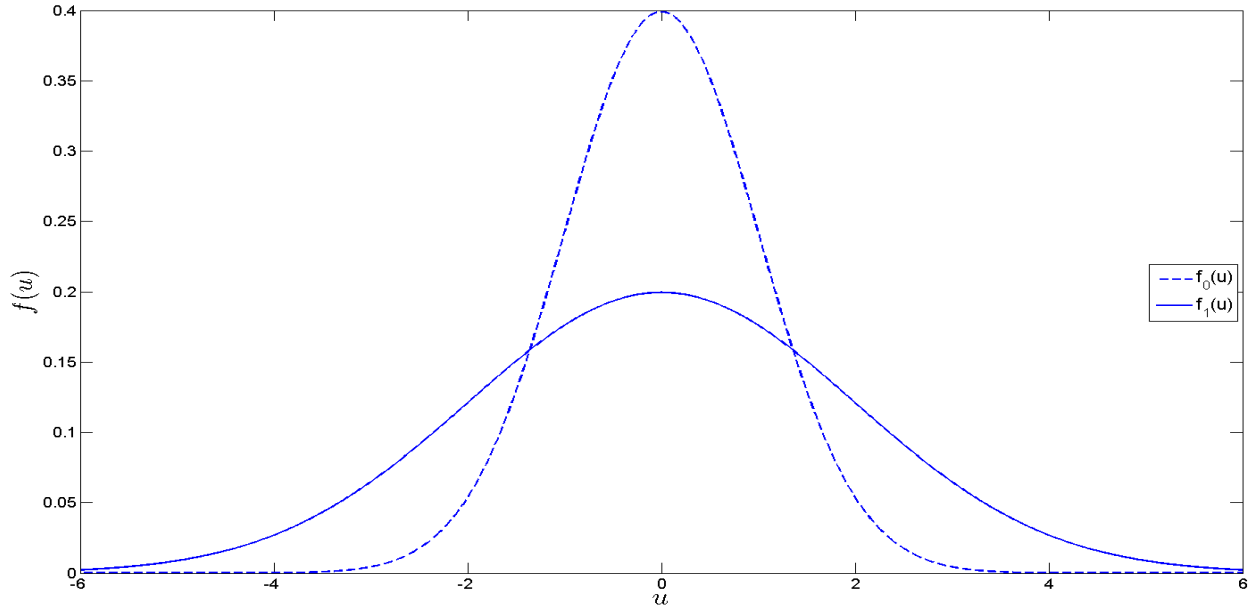
$$P_{\text{MD}} = \int_{0.2 \ln 4}^{\infty} 10 \cdot \exp(-10u) du = -\exp(-10u) \Big|_{0.2 \ln 4}^{\infty} = 0 - (-\exp(-2 \ln 4)) = \frac{1}{16}.$$

The average error probability thus is $\bar{P}_e = \frac{1}{3}P_{\text{FA}} + \frac{2}{3}P_{\text{MD}} = \frac{7}{24}$. Note that since $\pi_0 < \pi_1$, the Bayesian decision rule allows P_{FA} to increase in return for a decrease in P_{MD} because the latter is weighted more heavily.

(f) If the decision rule always decides H_1 is the true hypothesis it makes errors if and only if H_0 is the true hypothesis. Hence, $\bar{P}_e = \pi_0$.

(g) When $\pi_0 > 2/3$, the threshold ξ is less than 0. Since $\mathcal{X}$ takes on nonnegative values, it is always larger than the threshold, and hence the decision is always H_0 . The average error probability is π_1 , and since this is the minimum-error-probability rule, we cannot do any better than this. Note that $\pi_1 < 1/3$. When $\pi_0 > 2/3$, it follows that $\pi_0 > 2\pi_1$. The average probability of error for the maximum-likelihood rule is $\pi_0 \cdot (1/2) + \pi_1 \cdot (1/4) > 2\pi_1 \cdot (1/2) + \pi_1 \cdot (1/3) = 1.25\pi_1$.

6. The pdfs are sketched below. Note that it is immediately obvious that small values of $|\mathcal{X}|$ result in decisions in favor of H_0 and large values of $|\mathcal{X}|$ result in decisions in favor of H_1 .



(a) $\Lambda(u) = \frac{f_1(u)}{f_0(u)} = \frac{1/\sqrt{2\pi\sigma_1^2} \exp\{-u^2/2\sigma_1^2\}}{1/\sqrt{2\pi\sigma_0^2} \exp\{-u^2/2\sigma_0^2\}} = \frac{\sigma_0}{\sigma_1} \exp\left\{\frac{u^2}{2} \left[\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\right]\right\} = \frac{1}{\sqrt{2}} \exp\left(\frac{u^2}{4}\right)$ for the given values of σ_0^2 and σ_1^2 . The likelihood ratio has minimum value $1/\sqrt{2}$ at $u = 0$ and is an increasing function of $|u|$.

ML Decision Rule: Compare $\Lambda(u)$ to 1, or equivalently, compare $\ln \Lambda(u)$ to 0:

$$\begin{aligned}
& \ln \Lambda(u) \geq 0 \\
& \ln \left(\frac{\sigma_0}{\sigma_1} \exp \left\{ \frac{u^2}{2} \left[\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2} \right] \right\} \right) \geq 0 \\
& \ln \frac{\sigma_0}{\sigma_1} + \frac{u^2}{2} \left[\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2} \right] \geq 0 \\
& u^2 \geq \frac{2 \ln \left(\frac{\sigma_1}{\sigma_0} \right)}{\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}} \\
& |u| \geq \sqrt{\frac{2 \ln \left(\frac{\sigma_1}{\sigma_0} \right)}{\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}}} \\
& |u| \geq \sqrt{4 \ln \left(\sqrt{2} \right)} = \sqrt{2 \ln 2} = 1.1774 = \xi
\end{aligned}$$

So the ML decision rule becomes: **Choose H_1 if $|\mathcal{X}| \geq \xi$, otherwise choose H_0 .**

Bayes Decision Rule: Compare $\Lambda(u)$ to $\frac{\pi_0}{\pi_1}$, or equivalently, compare $\ln \Lambda(u) = \ln \frac{\sigma_0}{\sigma_1} + \frac{u^2}{2} \left[\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2} \right]$ to $\ln \frac{\pi_0}{\pi_1}$. The statement of the general rule is complicated because of the need for allowing for both possibilities: $\sigma_0^2 > \sigma_1^2$ and $\sigma_0^2 < \sigma_1^2$. For our specific problem with $\sigma_0^2 = 1, \sigma_1^2 = 2$,

$$\ln \frac{1}{\sqrt{2}} + \frac{u^2}{4} \geq \ln \frac{\pi_0}{\pi_1} \Rightarrow u^2 \geq 4 \left(\ln \sqrt{2} + \ln \frac{\pi_0}{\pi_1} \right).$$

Note that the right hand side is negative for $\pi_0 < \frac{1}{1+\sqrt{2}}$ and in this case, the decision is in favor of H_1 for *all* observed values of $\mathcal{X}$. For $\pi_0 \geq \frac{1}{1+\sqrt{2}}$, the decision is in favor of H_1 if $|u|$ exceeds

$$\sqrt{2 \ln 2 + 4 \ln(\pi_0/\pi_1)} = \sqrt{\xi + 4 \ln(\pi_0/\pi_1)}. \text{ Define } \gamma = \begin{cases} \sqrt{2 \ln 2 + 4 \ln(\pi_0/\pi_1)}, & \pi_0 \geq \frac{1}{1+\sqrt{2}}, \\ 0, & \pi_0 < \frac{1}{1+\sqrt{2}}. \end{cases}$$

So the Bayes decision rule becomes: **Choose H_1 if $|\mathcal{X}| \geq \gamma$, otherwise choose H_0 .**

Notice that $\gamma > \xi$ if $\pi_0 > \pi_1$; that is, because H_0 is the more likely hypothesis, the Bayes decision rule is playing the odds and making decisions in favor of H_0 even when $|\mathcal{X}|$ exceeds the ML threshold ξ by a little bit (but not by a lot). On the other hand, when $\pi_1/\pi_0 > \sqrt{2}$, the Bayes decision rule just ignores $\mathcal{X}$ and always chooses H_1 , the more likely hypothesis.

(b) Under H_0 , $X \sim \mathcal{N}(0, 1)$.

$$P_{\text{FA}} = P\{|X_0| \geq 1.1774\} = 1 - P\{|X_0| \leq 1.1774\} = 2(1 - \Phi(1.1774)) \approx 0.2390$$

Under H_1 , $X \sim \mathcal{N}(0, 2)$.

$$P_{\text{MD}} = P\{|X_1| \leq 1.1774\} = P\left\{-\frac{1.1774}{\sqrt{2}} \leq \frac{X}{\sqrt{2}} \leq \frac{1.1774}{\sqrt{2}}\right\} = 2\Phi(0.8325) - 1 \approx 0.5949$$