

## ECE 313: Solutions to Problem Set 9

1. (a)  $P\{\text{demand can be satisfied}\} = P\left\{\mathcal{X} \leq \frac{1}{2}\right\} = \int_0^{0.5} 5(1-u)^4 du = -(1-u)^5 \Big|_0^{0.5} = \frac{31}{32}.$
- (b) We want the minimum value of  $C$  such that  $P\{\mathcal{X} > C\} \leq 10^{-5}$ . But,
 
$$P\{\mathcal{X} > C\} = \int_C^1 5(1-u)^4 du = -(1-u)^5 \Big|_C^1 = (1-C)^5 \leq 10^{-5} \text{ if } C \geq 0.9.$$
- (c)  $\mathcal{Y} = \min\{\mathcal{X}, C\} = \begin{cases} \mathcal{X}, & \text{if } \mathcal{X} \leq C, \\ C, & \text{if } \mathcal{X} > C. \end{cases}$
- (d) The weekly *gross* profit is  $\$640\mathcal{Y}$  and the *average* weekly gross profit is

$$E[640\mathcal{Y}] = \int_0^C 5(1-u)^4(640u) du + \int_C^1 5(1-u)^4(640C) du = \frac{640}{6} [1 - (1-C)^6].$$

As a function of  $C$ , this increases from 0 at  $C = 0$  to  $640/6$  when  $C = 1$ . The average *net* profit is  $\$E[(640\mathcal{Y} - 20C)] = E[(640\mathcal{Y}) - 20C] = \frac{640}{6} [1 - (1-C)^6] - 20C$  whose maximum can be found by the usual calculus method to be at  $C = \frac{1}{2}$ . Thus, a 500 gallon tank that costs \$10 per week to rent gives an average gross profit of  $\$630/6$  and the maximum average net profit of  $\$570/6$  per week. In contrast, a full-sized 1000 gallon tank gives only a slightly larger gross profit of  $\$640/6$  per week, but a smaller net profit of  $\$520/6$  per week. Of course, there may be intangible losses in goodwill if the tank is emptied before the week ends. However, the probability of this happening is only  $\frac{1}{32}$  when  $C = \frac{1}{2}$ .

2. (a)  $E[\mathcal{Y}] = E[\mathcal{X}^2] = \int_{-1}^{+1} u^2 \left(\frac{1}{2}\right) du = \frac{1}{3}.$   $E[\mathcal{Y}^2] = E[\mathcal{X}^4] = \int_{-1}^{+1} u^4 \left(\frac{1}{2}\right) du = \frac{1}{5}.$   
 Hence,  $\text{var}(\mathcal{Y}) = E[\mathcal{Y}^2] - (E[\mathcal{Y}])^2 = \frac{1}{5} - \frac{1}{9} = \frac{4}{45}.$
- (b)  $E[\mathcal{Z}] = E[g(\mathcal{X})] = \int_{-1}^0 -u^2 \left(\frac{1}{2}\right) du + \int_0^{+1} u^2 \left(\frac{1}{2}\right) du = -\frac{1}{6} + \frac{1}{6} = 0.$
3. (a)  $P\{\mathcal{X} < 0\} = \Phi\left(\frac{0 - (-10)}{2}\right) = \Phi(5) = 1 - Q(5).$
- (b)  $P\{-10 < \mathcal{X} < 5\} = \Phi\left(\frac{5 - (-10)}{2}\right) - \Phi\left(\frac{-10 - (-10)}{2}\right) = \Phi(7.5) - \Phi(0) = \Phi(7.5) - \frac{1}{2}$   
 $= \frac{1}{2} - Q(7.5).$
- (c)  $P\{|\mathcal{X}| \geq 5\} = P\{\mathcal{X} \leq -5\} + P\{\mathcal{X} \geq 5\} = \Phi\left(\frac{-5 - (-10)}{2}\right) + 1 - \Phi\left(\frac{5 - (-10)}{2}\right)$   
 $= \Phi(2.5) + 1 - \Phi(7.5) = 1 - Q(2.5) + Q(7.5).$
- (d)  $P\{\mathcal{X}^2 - 3\mathcal{X} + 2 > 0\} = P\{(\mathcal{X} - 1)(\mathcal{X} - 2) > 0\} = P\{\mathcal{X} < 1\} + P\{\mathcal{X} > 2\}$   
 $= \Phi\left(\frac{1 - (-10)}{2}\right) + 1 - \Phi\left(\frac{2 - (-10)}{2}\right) = \Phi(5.5) + 1 - \Phi(6) = 1 - Q(5.5) + Q(6).$
4. Let  $\mathcal{X} \sim \mathcal{N}(0.9, 0.003^2)$  denote the width (in microns) of the trace.
  - (a)  $\{\mathcal{X} < 0.9 - 0.005\}$  or  $\{\mathcal{X} > 0.9 + 0.005\}$  for a trace to be deemed defective.  
 $P\{|\mathcal{X} - 0.9| > 0.005\} = 2\Phi(-0.005/0.003) = 2\Phi(-1.666\dots) = 2Q(1.666\dots) \approx 0.095.$
  - (b) We need to find the maximum value of  $\sigma$  such that  $2Q(0.005/\sigma) \leq 10^{-2}$ . Since  $Q(2.575) \approx 0.005$ , we get that  $0.0005/\sigma > 2.575$ , that is,  $\sigma \leq 0.005/2.575 \approx 0.00194.$

5. (a)  $\mathcal{I}$  can take on values in the range  $(-I_0, \infty)$ .  
 (b)  $F_{\mathcal{I}}(v) = 0$  for  $v < -I_0$ . For any  $v > -I_0$ ,  
 $F_{\mathcal{I}}(v) = P\{\mathcal{I} \leq v\} = P\{I_0(\exp(\mathcal{V}) - 1) \leq v\} = P\{\mathcal{V} \leq \ln(1 + v/I_0)\} = F_{\mathcal{V}}(\ln(1 + v/I_0))$ .  
 (c) The chain rule for differentiation gives that for  $v > -I_0$ ,

$$f_{\mathcal{I}}(v) = f_{\mathcal{V}}(\ln(1 + v/I_0)) \frac{1}{1 + v/I_0} \times \frac{1}{I_0} = \frac{f_{\mathcal{V}}(\ln(1 + v/I_0))}{v + I_0} = \begin{cases} \frac{I_0}{2(v + I_0)^2}, & v \geq 0, \\ \frac{1}{2I_0}, & -I_0 < v < 0, \end{cases}.$$

Note that the pdf has constant value  $1/(2I_0)$  from  $v = -I_0$  to  $v = 0$ .

6. (a) For  $k = 1, 2, 3, \dots$ ,  $P\{\mathcal{Y} = k\} = P\{\lceil \mathcal{X} \rceil = k\} = P\{k-1 < \mathcal{X} \leq k\} = F_{\mathcal{X}}(k) - F_{\mathcal{X}}(k-1)$ .  
 (b) When  $\mathcal{X}$  is an exponential random variable with parameter  $\lambda$ , then for  $k = 1, 2, 3, \dots$ ,  
 $P\{\mathcal{Y} = k\} = \int_{k-1}^k \lambda \exp(-\lambda u) du = -\exp(-\lambda u) \Big|_{k-1}^k = \exp(-\lambda(k-1)) - \exp(-\lambda k)$   
 $= \exp(-\lambda(k-1))[1 - \exp(-\lambda)] = (\exp(-\lambda))^{k-1}[1 - \exp(-\lambda)] = (1-p)^{k-1}p$  where  
 $p = 1 - \exp(-\lambda)$ . Hence,  $\mathcal{Y}$  is a geometric random variable with parameter  $p = 1 - \exp(-\lambda)$ .  
 (c) Since  $\mathcal{Y}$  is a geometric random variable with parameter  $p = 1 - \exp(-\lambda)$ , its expected value is

$$E[\mathcal{Y}] = \frac{1}{p} = \frac{1}{1 - \exp(-\lambda)} = \frac{1}{1 - (1 - \lambda + \lambda^2/2! - \dots)} = \frac{1}{\lambda - \lambda^2/2! + \dots} > \frac{1}{\lambda} = E[\mathcal{X}].$$

When  $\lambda$  is small,  $\lambda^2 \ll \lambda$  and hence  $E[\mathcal{Y}]$  is only slightly larger than  $E[\mathcal{X}]$ . This is to be expected since  $\mathcal{Y} = \lceil \mathcal{X} \rceil$  is just a little larger than  $\mathcal{X}$ . On the other hand, when  $\lambda \gg 1$ , most of the probability mass is close to the origin which is also suggested by the fact that  $E[\mathcal{X}] = \lambda^{-1}$  is close to 0. Thus, with high probability  $\mathcal{X} < 1$ , and therefore with high probability,  $\mathcal{Y} = \lceil \mathcal{X} \rceil = 1$ . Thus, intuitively, one would expect that  $E[\mathcal{Y}]$  is only slightly larger than 1, and this is confirmed by the mathematics.  $\exp(-\lambda)$  is very small when  $\lambda$  is large and hence  $E[\mathcal{Y}] = (1 - \exp(-\lambda))^{-1}$  is only slightly larger than 1.

In engineering terms, we need to use a higher clock rate. Otherwise, the recorded time of arrival of the radar echo is a gross mis-representation of the actual time of arrival.

7. (a) The pmf of  $\mathcal{Y}$  is  $p_{\mathcal{Y}}(\alpha) = p_{\mathcal{Y}}(-\alpha) = \frac{1}{2}$ .  
 (b)  $E[\mathcal{Z}] = \int_0^\infty (u-\alpha)^2 \phi(u) du + \int_{-\infty}^0 (u+\alpha)^2 \phi(u) du = \int_{-\infty}^\infty (u^2 + \alpha^2) \phi(u) du - 4 \int_0^\infty \alpha u \phi(u) du$   
 $= 1 + \alpha^2 - 2\sqrt{\frac{2}{\pi}}\alpha$  where we have used the facts that the standard Gaussian random variable has variance 1, the area under the pdf  $\phi(u)$  is 1, and  $\int_0^\infty u \exp(-u^2/2) du = 1$  (cf. Problem 6(b) of Problem Set 1) in arriving at the result.  $E[\mathcal{Z}]$  has minimum value  $1 - \frac{2}{\sqrt{\pi}}$  at  $\alpha = \sqrt{2/\pi}$ .  
 (c)  $p_{\mathcal{W}}(3) = p_{\mathcal{W}}(-3) = \Phi(-2.5) = 0.0062$ .  $p_{\mathcal{W}}(2) = p_{\mathcal{W}}(-2) = \Phi(2.5) - \Phi(1.5) = 0.0606$ .  
 $p_{\mathcal{W}}(1) = p_{\mathcal{W}}(-1) = \Phi(1.5) - \Phi(0.5) = 0.2417$ .  $p_{\mathcal{W}}(0) = \Phi(0.5) - \Phi(-0.5) = 0.3830$ .  
 (d)  $\mathcal{Z}_2, \mathcal{Z}_1, \mathcal{Z}_0$  are *Bernoulli* random variables with parameters  $p_2 = P\{\mathcal{W} < 0\} = 0.3085$ ,  
 $p_1 = P\{\mathcal{W} \in \{-2, -1, 2, 3\}\} = 0.3691$ , and  $p_0 = P\{\mathcal{W} \in \{-3, -1, 1, 3\}\} = 0.4958$  respectively.