

ECE 313: Solutions to Problem Set 7

1. (a) D denotes the event that a *majority* predicts a pass in ECE 313.

$$\begin{aligned}
 P(D) &= P((A \cap B) \cup (B \cap C) \cup (A \cap C)) \\
 &= P(A \cap B) + P(A \cap C) + P(B \cap C) - 2P(A \cap B \cap C) \\
 &= P(A)P(B) + P(A)P(C) + P(B)P(C) - 2P(A)P(B)P(C) \quad \text{by independence} \\
 &= \left(\frac{1}{3}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{3}\right)\left(\frac{3}{4}\right) + \left(\frac{1}{2}\right)\left(\frac{3}{4}\right) - 2 \times \left(\frac{1}{3}\right)\left(\frac{1}{2}\right)\left(\frac{3}{4}\right) = \frac{13}{24}.
 \end{aligned}$$

$$(b) P(A^c|D) = \frac{P(A^c \cap D)}{P(D)} = \frac{P(A^c \cap B \cap C)}{13/24} = \frac{(2/3) \times (1/2) \times (3/4)}{13/24} = \frac{6}{13}.$$

2. (a) Yes. Clearly the outcomes of the first and third tosses do not depend on each other.
- (b) A and C are stochastically independent: $P(AC) = P(A)P(C) = \frac{1}{8}$, but not physically independent. If $P(H) = p$, $P(AC) = p^2(1-p)$, but $P(A)P(C) = 2p^3(1-p) \neq p^2(1-p)$ except when $p = \frac{1}{2}$ or in the trivial cases $p = 0$ and $p = 1$.
- (c) B and C are stochastically independent: $P(BC) = P(A)P(C) = \frac{1}{8}$ but not physically independent. If $P(H) = p$, $P(BC) = p^2(1-p)$, but $P(B)P(C) = 2p^3(1-p) \neq p^2(1-p)$ except when $p = \frac{1}{2}$ or in the trivial cases $p = 0$ and $p = 1$.
- (d) No. A, B, C are not independent events. $ABC = \emptyset$ and hence $P(ABC) = 0 \neq P(A)P(B)P(C)$ except in the trivial cases $p = 0$ and $p = 1$.
- (e) Yes, the events A, B, C are (stochastically) pairwise independent when $p = \frac{1}{2}$ (or in the trivial cases $p = 0$ and $p = 1$) as per the answers to parts (a), (b), and (c).
3. (a) The probability of getting a left shoe on the first draw is $\frac{1}{2}$, and given this, the probability of getting a right shoe on the second draw is $\frac{13}{25}$. Hence, the probability of getting a left shoe on the first draw and a right shoe on the second draw is $\frac{1}{2} \times \frac{13}{25} = \frac{13}{50}$. Similarly, the probability that the first shoe is right and the second shoe is left is $\frac{13}{50}$. The probability of two left shoes is $\frac{1}{2} \times \frac{12}{25} = \frac{6}{25}$. The probability of two right shoes is $\frac{1}{2} \times \frac{12}{25} = \frac{6}{25}$. The probabilities do add to 1. What other outcome would be possible?
- (b) Let $P(A_n)$ be the probability of not drawing a match on draw n . The probability of no matching shoes is
 $P(A_1 A_2 A_3 A_4) = P(A_4|A_1 A_2 A_3)P(A_3|A_1 A_2)P(A_2|A_1)P(A_1) = \frac{20}{23} \times \frac{22}{24} \times \frac{24}{25} \times 1 = \frac{88}{115}$.
- (c) The event “four left shoes” is a subset of the event “no matching pair” and hence $P(\text{four left shoes} \mid \text{no matching pair}) = P(\text{four left shoes})/P(\text{no matching pair})$.
 But, $P(\text{four left shoes}) = \frac{13}{26} \times \frac{12}{25} \times \frac{11}{24} \times \frac{10}{23}$ while from part (b),
 $P(\text{no matching pair}) = \frac{26}{26} \times \frac{24}{25} \times \frac{22}{24} \times \frac{20}{23}$.
 Hence, $P(\text{four left shoes} \mid \text{no matching pair}) = \frac{1}{16}$.
4. (a) Consider the contents of the first two boxes.
- If the first two boxes give us one S and one H (this has probability $\frac{3}{8}$), then in effect we are looking at the Hour Exam I problem, since we need to collect one H and one S . Thus, conditioned on the first two boxes having an H and a S , $\mathcal{Y} = 2 + \mathcal{X}$ where $\mathcal{X}$ was considered on Hour Exam I. Recall that $E[\mathcal{X}] = 4\frac{1}{3}$ as found on the Exam.

- If the first two boxes give us two H 's (probability $\frac{9}{16}$) or two S 's (probability $\frac{1}{16}$), then we have to wait for two S 's (or two H 's) to occur. Conditioned on this event HH (or SS), $\mathcal{Y} = 2 + \mathcal{V}$ where $\mathcal{V}$ is a *negative binomial* random variable with parameters $(2, \frac{1}{4})$ (or $(2, \frac{3}{4})$) and mean 8 (or $\frac{8}{3}$).

$$\text{Hence, for } k \geq 4, P\{\mathcal{Y} = k\} = P\{\mathcal{X} = k-2\} \frac{3}{8} + P\{\mathcal{V} = k-2\} \frac{9}{16} + P\{\mathcal{V} = k-2\} \frac{1}{16}$$

$$= \left[\left(\frac{1}{4}\right) \left(\frac{3}{4}\right)^{k-3} + \left(\frac{3}{4}\right) \left(\frac{1}{4}\right)^{k-3} \right] \frac{3}{8} + \left[(k-3) \left(\frac{1}{4}\right)^2 \left(\frac{3}{4}\right)^{k-4} \right] \frac{9}{16} + \left[(k-3) \left(\frac{3}{4}\right)^2 \left(\frac{1}{4}\right)^{k-4} \right] \frac{1}{16}$$

Computing $E[\mathcal{Y}]$ from this result is messy. It is easier to use $E[\mathcal{Y}] = \sum_{k=0}^{\infty} P\{\mathcal{Y} > k\}$ where we know that $P\{\mathcal{Y} > k\} = 1$ for $k = 0, 1, 2$, and 3 . (Why?) For $k \geq 4$, the event $\{\mathcal{Y} > k\}$ occurs if and only if the k boxes contain *at most* one H or *at most* one S . Hence, $P\{\mathcal{Y} > k\} = \left[\left(\frac{1}{4}\right)^k + k \left(\frac{1}{4}\right)^{k-1} \frac{3}{4} \right] + \left[\left(\frac{3}{4}\right)^k + k \left(\frac{3}{4}\right)^{k-1} \frac{1}{4} \right]$ for $k \geq 4$ and thus

$$E[\mathcal{Y}] = \sum_{k=0}^{\infty} P\{\mathcal{Y} > k\} = 4 + \sum_{k=4}^{\infty} \left[\left(\frac{1}{4}\right)^k + k \left(\frac{1}{4}\right)^{k-1} \frac{3}{4} \right] + \left[\left(\frac{3}{4}\right)^k + k \left(\frac{3}{4}\right)^{k-1} \frac{1}{4} \right].$$

$$\text{But, } \sum_{k=4}^{\infty} x^k = x^4 [1 + x + x^2 + \dots] = \frac{x^4}{1-x} \text{ giving } \sum_{k=4}^{\infty} \left(\frac{1}{4}\right)^k = \frac{1}{192}, \quad \sum_{k=4}^{\infty} \left(\frac{3}{4}\right)^k = \frac{243}{192}$$

$$\text{while } \sum_{k=4}^{\infty} kx^{k-1} = 4x^3 + 5x^4 + \dots = [1 + 2x + 3x^2 + 4x^3 + \dots] - 1 - 2x - 3x^2$$

$$= \frac{1}{(1-x)^2} - 1 - 2x - 3x^2 \text{ giving } \sum_{k=4}^{\infty} k \left(\frac{1}{4}\right)^{k-1} \frac{3}{4} = \frac{13}{192} \text{ and } \sum_{k=4}^{\infty} k \left(\frac{3}{4}\right)^{k-1} \frac{1}{4} = \frac{567}{192} \text{ and}$$

$$\text{thus } E[\mathcal{Y}] = \frac{1 + 243 + 13 + 567}{192} = 8\frac{7}{24}.$$

It is even more easy to compute $E[\mathcal{Y}]$ via conditional probabilities. If the first two boxes give a H and a S , we need $4\frac{1}{3}$ *more* boxes on average. If the first two boxes are HH , 8 *more* boxes are needed on average, while if the first two boxes are SS , $\frac{8}{3}$ *more* boxes are needed on average. Hence, $E[\mathcal{Y}] = 2 + 4\frac{1}{3} \times \frac{3}{8} + 8 \times \frac{9}{16} + \frac{8}{3} \times \frac{1}{16} = 8\frac{7}{24}$ as before.

5. (a) $P(F_1) = \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n = \frac{2}{3}$. $P(W_1) = 1 - P(F_1) = \frac{1}{3}$
 $P(F_2) = P(F_2|F_1)P(F_1) + P(F_2|W_1)P(W_1) = \frac{1}{3} \times \frac{2}{3} + \frac{2}{3} \times \frac{1}{3} = \frac{4}{9}$. $P(W_2) = \frac{5}{9}$
- (b) $P(F_{n+1}) = \frac{1}{3}P(F_n) + \frac{2}{3}P(W_n) = \frac{1}{3}P(F_n) + \frac{2}{3}(1 - P(F_n)) = \frac{2}{3} - \frac{1}{3}P(F_n)$.
 $P(W_{n+1}) = \frac{2}{3}P(F_n) + \frac{1}{3}P(W_n) = \frac{2}{3}(1 - P(W_n)) + \frac{1}{3}P(W_n) = \frac{2}{3} - \frac{1}{3}P(W_n)$.
- (c) Suppose that $P(F_n) = a + b\alpha^n$ for $n \geq 1$. Then we have that

$$a + b\alpha^{n+1} = \frac{2}{3} - \frac{1}{3} \left(a + b\alpha^n \right) \Rightarrow b\alpha^n \left(\alpha + \frac{1}{3} \right) = \frac{2-4a}{3} \text{ for all } n \geq 1.$$

In order for this to hold for all $n > 1$, it must be that $\alpha = -\frac{1}{3}$ and $a = \frac{1}{2}$. Hence,

$$P(F_n) = \frac{1}{2} + b \left(-\frac{1}{3} \right)^n$$

and since $P(F_1) = \frac{2}{3}$, we get that $b = -\frac{1}{2}$. The calculation for $P(W_n)$ is almost identical, except that the initial condition $P(W_1) = \frac{1}{3}$ gives a different value $+\frac{1}{2}$ for b . Hence, we get that

$$P(F_n) = \frac{1}{2} \left[1 - \left(\frac{-1}{3} \right)^n \right] \text{ and } P(W_n) = \frac{1}{2} \left[1 + \left(\frac{-1}{3} \right)^n \right].$$

Note that $P(F_n) > \frac{1}{2} > P(W_n)$ for odd n , and $P(F_n) < \frac{1}{2} < P(W_n)$ for even n .

(d) Since $(\frac{1}{3})^n \rightarrow 0$ as $n \rightarrow \infty$, $\lim_{n \rightarrow \infty} P(F_n) = \lim_{n \rightarrow \infty} P(W_n) = \frac{1}{2}$ and the game is asymptotically fair.

6. (a) The bill passes if the Conservative Republicans and at least one other group votes for it, i.e., the event $A \cap (B \cup C \cup D)$ occurs, or if all groups except the Conservative Republicans vote for it, i.e., the event $\{A^c \cap B \cap C \cap D\}$ occurs. Note that these are disjoint events (Why?). Now, $(B \cup C \cup D)^c = B^c \cap C^c \cap D^c$ by DeMorgan's theorem, and thus we can partition A into two disjoint events as follows: $A = A \cap (B \cup C \cup D) \cup (A \cap (B^c \cap C^c \cap D^c))$. Hence, we see that

$$P(A \cap (B \cup C \cup D)) = P(A) - P(A \cap B^c \cap C^c \cap D^c) = P(A) - P(A)P(B^c)P(C^c)P(D^c)$$

by independence. Since $P(A^c \cap B \cap C \cap D) = P(A^c)P(B)P(C)P(D)$ by independence, we get

$$\begin{aligned} P(\text{bill passes}) &= P(A \cap (B \cup C \cup D)) + P(A^c \cap B \cap C \cap D) \\ &= P(A) - P(A)P(B^c)P(C^c)P(D^c) + P(A^c)P(B)P(C)P(D) \\ &= 0.9 \times [1 - 0.4 \times 0.5 \times 0.8] + 0.1 \times 0.6 \times 0.5 \times 0.2 \\ &= 0.756 + 0.006 = 0.762 \end{aligned}$$

- (b) The Conservative Republicans must be in favor for the motion to override to pass. Furthermore, since politics makes strange bedfellows, either the Liberal Democrats must also be in favor or both the other groups must also be in favor. Hence,

$$\begin{aligned} P(\text{veto overridden}) &= P(E \cap (H \cup (F \cap G))) \\ &= P(E) \times P(H \cup (F \cap G)) \quad \text{by independence of } E \text{ and } H \cup (F \cap G) \\ &= P(E) \times [P(H) + P(F \cap G) - P(F \cap G \cap H)] \\ &= P(E) \times [P(H) + P(F)P(G) - P(F)P(G)(H)] \quad \text{independence again!} \\ &= 0.99 \times [0.1 + 0.4 \times 0.6 - 0.6 \times 0.4 \times 0.1] \\ &= 0.31284 \end{aligned}$$

7. Let $q = 1 - p$ be the probability that a link is in working condition.

- (a) Suppose that the VIVA LET to NAMMA link has failed. Then, it is possible to call from ORIAC to SUCSAMAD if both the ORIAC to ZEUS and NONABEL to SUCSAMAD link are working *and at least one* of the two paths from ZEUS to NONABEL is working. The probability of this is $q \cdot q \cdot (2q^2 - (q^2)^2) = 2q^4 - q^6$. On the other hand, if the VIVA LET to NAMMA link is working, then it is possible to call from ORIAC to SUCSAMAD if both the ORIAC to ZEUS and NONABEL to SUCSAMAD link are working and *at least one* of the two paths eastwards out of ZEUS as well as at least one of the two paths eastwards in to NONABEL is working. The probability of this is $q \cdot q \cdot (2q - q^2)^2 = 4q^4 - 4q^5 + q^6$. Hence, by the theorem of total probability, the probability of being able to call from ORIAC to SUCSAMAD is

$$(2q^4 - q^6)(1 - q) + (4q^4 - 4q^5 + q^6)q = q^4(2 + 2q - 5q^2 + 2q^3) = q^4(1 + 2p + p^2 - 2p^3).$$

- (b) The probability of being able to call from ORIAC to SUCSAMAD *and* the ZEUS to NAMMA link being *failed* is $p \cdot q^3 \cdot (q + q^2 - q^3)$ since either the VIVA LET to NONABEL link or the VIVA LET to NONABEL via NAMMA path must be in working condition. Hence,

$$P\{\text{Z-N OK} | \text{O-C OK}\} = 1 - \frac{pq^4(1 + q^2 - q^3)}{q^4(1 + 2p + p^2 - 2p^3)} = \frac{p(1 + 2p - 2p^2)}{1 + 2p + p^2 - 2p^3}.$$

- (c) $\mathcal{X}$ is nonnegative, and has a maximum value of 70 exactly when all links are in working condition. This has probability q^7 , i.e., $p_{\mathcal{X}}(70) = q^7$.

We can also easily get

$$p_{\mathcal{X}}(0) = 1 - P(\text{path from ORIAC to SUCSAMAD}) = 1 - q^4(1 + 2p + p^2 - 2p^3).$$

If at least one of the ZEUS to VIVALET and NAMMA to NONABEL links (which have capacity 20) has failed while all other five links are working, then $\mathcal{X} = 50$ and $p_{\mathcal{X}}(50) = q^5(2p - p^2)$.

If both the ORIAC to ZEUS link and the NONABEL to SUCSAMAD link are in working condition, but the VIVA LET to NAMMA link has failed, then $\mathcal{X} = 40$ if *both* paths from ZEUS to NONABEL are working. We thus have $p_{\mathcal{X}}(40) = q^2 \cdot p \cdot q^2 \cdot q^2 = q^6p$.

The only other possibility is that $\mathcal{X} = 20$, and we can most easily find $p_{\mathcal{X}}(20)$ as

$$\begin{aligned} p_{\mathcal{X}}(20) &= 1 - p_{\mathcal{X}}(0) - p_{\mathcal{X}}(40) - p_{\mathcal{X}}(50) - p_{\mathcal{X}}(70) \\ &= q^4(1 + 2p + p^2 - 2p^3) - q^6p - q^5(2p - p^2) - q^7. \end{aligned}$$

Alternatively, if both the ORIAC to ZEUS link and the NONABEL to SUCSAMAD link are in working condition, but the VIVA LET to NAMMA link has failed, and *exactly one* of the paths from ZEUS to NONABEL is working, then $\mathcal{X} = 20$. This has probability $q^2 \cdot p \cdot 2 \cdot q^2 \cdot (2p - p^2) = 2q^4p^2(2 - p) = 2q^4p^2(1 + q) = 2q^4p^2 + 2q^5p^2 = 2q^4p^2(p + q) + 2q^5p^2 = 2q^4p^3 + 4q^5p^2$.

We also have $\mathcal{X} = 20$ if at least one of the two links of capacity 50 has failed, but everything else is in working condition (probability $q^5(2p - p^2) = q^5p^2 + 2q^6p$), or five links are working and two links comprising a path from ZEUS to NONABEL have failed (probability $2q^5p^2$). Hence,

$$\begin{aligned} p_{\mathcal{X}}(20) &= 2q^4p^3 + 4q^5p^2 + q^5p^2 + 2q^6p + 2q^5p^2 \\ &= 2q^4p^3 + 7q^5p^2 + 2q^6p. \end{aligned}$$

Hence, $E[\mathcal{X}] = 20(2q^4p^3 + 7q^5p^2 + 2q^6p) + 40(q^6p) + 50(q^5(2p - p^2)) + 70(q^7)$. As a simple check, note that $E[\mathcal{X}] = 70$ if $p = 0$ and 0 if $p = 1$.

8. (a) For the single gigantic car, each part fails with probability p . All the N parts of a given type must fail in order for the car to fail, and this occurs with probability p^N , and so the probability that at least one part of a given type is working is $1 - p^N$. The probability that at least one part of each of the M types is working is $(1 - p^N)^M$, and hence the probability of system failure is $1 - (1 - p^N)^M$.

On the other hand, if we have N cars each with M parts, then the probability that at least one part fails in a car is $1 - (1 - p)^M$. This is also the probability that the car itself has failed. But, we have N cars, and thus the probability that at least one car is in working condition (i.e. that not all have failed) is $(1 - (1 - p)^M)^N$.

- (b) For the single gigantic car, we want $1 - ((1 - 0.2^N)^5) < 10^{-3}$ which is achieved by choosing $N \geq 6$. For the N small cars, we want $(1 - (1 - 0.2^5)^N) < 10^{-3}$ which is achieved by choosing $N \geq 18$. Thus, a gigantic car with 6 engines, 6 transmissions, 6 brakes etc provides more reliable transmission than having 17 different compact cars.
- (c) If $M = 1000$, we get that $N \geq 9$ for the gigantic car while typical calculators blow up on the calculation for the number of small cars needed. The Unix high-precision utility *bc* gives that $1 - (1 - 0.2)^{1000} = 0.99 \dots 9986160344 \dots$ where there are eightyseven 9's preceding the 8616, and the number of cars needed is astronomical. Parking, of course, is impossible, but keep in mind that a gigantic car with 9 engines etc will also be very difficult to park!

Moral: It is better to replicate parts than to replicate systems.