

ECE 313: Solutions to Problem Set 6

1. Let
- X
- = bit stored,
- Y
- = bit read

$$(a) P(Y = 1) = P(Y = 1|X = 1)P(X = 1) + P(Y = 1|X = 0)P(X = 0) \\ = (0.85)(0.5) + (0.1)(0.5) = 0.475.$$

$$(b) P(X = 1|Y = 1) = \frac{P(Y=1, X=1)}{P(Y=1)} = \frac{P(X=1)P(Y=1|X=1)}{P(Y=1)} = \frac{(0.5)(0.85)}{0.475} = 0.8947.$$

2. (a)
- $P(T+) = P(T+|H+)P(H+) + P(T+|H-)P(H-).$

$$\text{Test A: } P(T+) = (0.999)(0.02) + (0.01)(0.98) = 0.02978.$$

$$\text{Test B: } P(T+) = (0.99)(0.02) + (0.001)(0.98) = 0.02078.$$

$$(b) P(H+|T+) = \frac{P(H+, T+)}{P(T+)} = \frac{P(H+)P(T+|H+)}{P(T+)}.$$

$$\text{Test A: } P(H+|T+) = \frac{(0.02)(0.999)}{(0.02978)} = 0.67092.$$

$$\text{Test B: } P(H+|T+) = \frac{(0.02)(0.99)}{(0.02078)} = 0.952839.$$

$$(c) P(H+|T-) = \frac{P(H+)P(T-|H+)}{P(T-)} = \frac{P(H+)[1-P(T+|H+)]}{[1-P(T+)]}.$$

$$\text{Test A: } P(H+|T-) = \frac{(0.98)(1-0.999)}{(1-0.2978)} = 0.00101.$$

$$\text{Test B: } P(H+|T-) = \frac{(0.98)(1-0.99)}{(1-0.2078)} = 0.010008.$$

3. (a) Note that each row of the matrix (call it
- A
-) is just the
- conditional*
- pmf of
- $\mathcal{Y}$
- conditioned on the value of
- $\mathcal{X}$
- . Let
- $\vec{P}_{\mathcal{X}} = [0.5, 0.25, 0.25]$
- be the
- pmf vector*
- for
- $\mathcal{X}$
- . Then,
- $\vec{P}_{\mathcal{Y}} = \vec{P}_{\mathcal{X}} A$
- . More specifically,
- $P\{\mathcal{Y} = j\} = \sum_{i=1}^3 P\{\mathcal{Y} = j|\mathcal{X} = i\}P\{\mathcal{X} = i\}$

$$= \begin{cases} 0.8 \times 0.5 + 0.05 \times 0.25 + 0.15 \times 0.25 & = 0.45 & \text{for } j = 1 \\ 0.1 \times 0.5 + 0.9 \times 0.25 + 0.05 \times 0.25 & = 0.2875 & \text{for } j = 2 \\ 0.1 \times 0.5 + 0.05 \times 0.25 + 0.8 \times 0.25 & = 0.2625 & \text{for } j = 3 \end{cases}$$

Sanity check: the three probabilities sum to 1.

$$(b) P\{\mathcal{X} = i|\mathcal{Y} = 3\} = \frac{P\{\mathcal{Y} = 3|\mathcal{X} = i\}P\{\mathcal{X} = i\}}{P\{\mathcal{Y} = 3\}} = \begin{cases} \frac{0.1 \times 0.5}{0.2625} & = \frac{4}{21} & \text{for } i = 1, \\ \frac{0.05 \times 0.25}{0.2625} & = \frac{1}{21} & \text{for } i = 2, \\ \frac{0.8 \times 0.25}{0.2625} & = \frac{16}{21} & \text{for } i = 3. \end{cases}$$

This is the conditional pmf of $\mathcal{X}$ given that $\mathcal{Y}$ was observed to have value 3. Note that the sum of the three probabilities is 1, as it should be for a valid pmf.

4. (a)
- $P(F_1) = P(F_1|E_1)P(E_1) + P(F_1|E_2)P(E_2) + P(F_1|E_3)P(E_3) + P(F_1|E_4)P(E_4)$
-
- $= (0.9)(0.4) + (0)(0.3) + (0)(0.2) + (0)(0.1) = 0.36$

- (b) If the MS was found on the first page, it implies that the MS is located in cell 1
- $\Rightarrow P(E_1|F_1) = 1$
- .

- (c) No matter which cell the MS is located in, the first time the MS is paged in the cell in which it is located, it will be missed with probability 0.1. Given that the MS is missed in the first round, it will be missed again in the second round with conditional probability 0.1. So
- $P(G_8) = (0.1)(0.1) = 0.01$
- .

- (d) Now
- $P(F_2|G_1) = \frac{P(F_2G_1)}{P(G_1)}$
- . By the law of total probability with the partition consisting of events
- E_1, E_2, E_3, E_4
- ,

$$P(G_1) = P(G_1|E_1)P(E_1) + P(G_1|E_2)P(E_2) + P(G_1|E_3)P(E_3) + P(G_1|E_4)P(E_4) \\ = (0.1)(0.4) + (1)(0.3) + (1)(0.2) + (1)(0.1) = 0.64$$

and

$$P(F_2G_1) = P(F_2G_1|E_1)P(E_1) + P(F_2G_1|E_2)P(E_2) + P(F_2G_1|E_3)P(E_3) \\ + P(F_2G_1|E_4)P(E_4) \\ = (0)(0.4) + (0.9)(0.3) + (0)(0.2) + (0)(0.1) = 0.27$$

Thus, $P(F_2|G_1) = \frac{0.27}{0.64} \approx 0.4219$.

(e) We are seeking $P(F_4|G_3) = \frac{P(F_4G_3)}{P(G_3)}$, where

$$\begin{aligned} P(G_3) &= P(G_3|E_1)P(E_1) + P(G_3|E_2)P(E_2) + P(G_3|E_3)P(E_3) \\ &\quad + P(G_3|E_4)P(E_4) \\ &= (0.1)(0.4) + (0.1)(0.3) + (0.1)(0.2) + (1)(0.1) = 0.19 \end{aligned}$$

and

$$\begin{aligned} P(F_4G_3) &= P(F_4G_3|E_1)P(E_1) + P(F_4G_3|E_2)P(E_2) + P(F_4G_3|E_3)P(E_3) \\ &\quad + P(F_4G_3|E_4)P(E_4) \\ &= (0)(0.4) + (0)(0.3) + (0)(0.2) + (0.9)(0.1) = 0.09 \end{aligned}$$

Thus, $P(F_4|G_3) = \frac{0.09}{0.19} \approx 0.4737$.

(f) We are seeking $P(F_8|G_7) = \frac{P(F_8G_7)}{P(G_7)}$, where

$$\begin{aligned} P(G_7) &= P(G_7|E_1)P(E_1) + P(G_7|E_2)P(E_2) + P(G_7|E_3)P(E_3) \\ &\quad + P(G_7|E_4)P(E_4) \\ &= (0.01)(0.4) + (0.01)(0.3) + (0.01)(0.2) + (0.1)(0.1) = 0.019 \end{aligned}$$

and

$$\begin{aligned} P(F_8G_7) &= P(F_8G_7|E_1)P(E_1) + P(F_8G_7|E_2)P(E_2) + P(F_8G_7|E_3)P(E_3) \\ &\quad + P(F_8G_7|E_4)P(E_4) \\ &= (0)(0.4) + (0)(0.3) + (0)(0.2) + (0.09)(0.1) = 0.009 \end{aligned}$$

Thus, $P(F_8|G_7) = \frac{0.009}{0.019} \approx 0.4737$.

Note: An explanation for why the answer to (d) is the same as the answer to (c) is that if the MS is not found after the first round of four pages, the conditional distribution of which cell the MS is located in is the same as the original distribution.

5. $P(\text{guilty}) = 0.8$. $P(\text{Bolies}|\text{guilty}) = 0.2$. $P(\text{Clarialies}|\text{innocent}) = 0.3$.

- (a) $P(\text{Bolies}) = P(\text{Bolies}|\text{guilty})P(\text{guilty}) + P(\text{Bolies}|\text{innocent})P(\text{innocent})$
 $= (0.2)(0.8) + (0)(0.2) = 0.16$.
 $P(\text{Clarialies}) = P(\text{Clarialies}|\text{guilty})P(\text{guilty}) + P(\text{Clarialies}|\text{innocent})P(\text{innocent})$
 $= (0)(0.8) + (0.3)(0.2) = 0.06$
 So Clara is more likely to testify truthfully.

- (b) Conflicting testimony results when either Bo or Clara lies, but the other tells the truth. Since Bo lies only when Adam is guilty and Clara lies only when Adam is innocent, these events are mutually exclusive, and we can simply add the probabilities:

$$P(\text{conflict}) = P(\text{Bolies}) + P(\text{Clarialies}) = 0.22$$

- (c) $P(\text{innocent}|\text{conflict}) = \frac{P(\text{innocent, conflict})}{P(\text{conflict})} = \frac{P(\text{innocent})P(\text{conflict}|\text{innocent})}{P(\text{conflict})}$
 $= \frac{P(\text{innocent})P(\text{claralies}|\text{innocent})}{P(\text{conflict})} = \frac{(0.2)(0.3)}{0.22} = 0.2727$

6. $P(A) = P(B) = \frac{1}{2}$. $P(C) = (\frac{1}{2})(\frac{1}{2})^2 \binom{2}{1} = \frac{1}{4}$

- (a) Yes. $P(AC) = P(A)P(C|A) = (\frac{1}{2})(\frac{1}{2})^2 = \frac{1}{8} = P(A)P(C)$.
 (b) No. $P(BC) = P(B)P(C|B) = (\frac{1}{2})(\frac{1}{2})^2 \binom{2}{1} = \frac{1}{4} \neq P(B)P(C) = \frac{1}{8}$.
 (c) No. B and C are not independent.
 (d) No. B and C are not independent.